

# FERMIONIC QUANTUM OPERATIONS: A COMPUTATIONAL FRAMEWORK I. BASIC INVARIANCE PROPERTIES

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**ABSTRACT.** The objective of this series of papers is to recover information regarding the behaviour of FQ operations in the case  $n = 2$ , and FQ conform-operations in the case  $n = 3$ . In this first part we study how the basic invariance properties of FQ operations ( $n = 2$ ) are reflected in their formal power series expansions.

## INTRODUCTION

Dealing with functions of several noncommuting operators has many approaches, like holomorphic calculus around the joint spectrum [6], operator ordering [5], using Clifford variables [1], integrated functional calculi [2], advanced perturbation techniques and rational noncommutative functions [3] (also see references therein). Here we intend to investigate a very specific one, basically considering functions defined on perturbations of Clifford systems: FQ (conform-)operations were introduced by the author in [4] as a kind of non-commutative linear algebra and/or functional calculus. These operation have both analytic and algebraic aspects, here we center on some effective computational techniques regarding them, primarily from algebraic viewpoint. The objective of this series of papers is to recover information regarding the behaviour of FQ operations in the case  $n = 2$ , and FQ conform-operations in the case  $n = 3$ . In this first part we study how the basic invariance properties of FQ operations ( $n = 2$ ) are reflected in their formal power series expansions. We concentrate on formal FQ operations, and, for the sake of simplicity, only on ones with good sign-linear properties. On the other hand, although our ultimate objective is the study of natural FQ operations, we will typically consider expansions around a fixed Clifford system  $(Q_1, Q_2)$  without the a priori assumption of naturality / conjugation invariance.

## 1. FQ OPERATIONS: EXPANSIONS AND BASES

1.1. Following [4], formal FQ operations can be represented as formal power series expansions around Clifford systems. In the case  $n = 2$ , we have a Clifford system  $(Q_1, Q_2)$ , and then the FQ operation is considered to make sense for the pair  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$ , where  $R_1$  and  $R_2$  are imagined as infinitesimal or formal variables.

If  $Q$  is a skew-involution, then we use notation

$$R_Q^0 := \frac{1}{2} (R + Q^{-1} R Q), \quad R_Q^1 := \frac{1}{2} (R - Q^{-1} R Q);$$

and

$$(R/Q)_j^{(\iota_1, \iota_2)} := (R_j Q_j^{-1})_{Q_1 Q_2}^{\iota_1 \iota_2}.$$

Now, a scalar FQ operation with property (sL) can be written as

$$(1) \quad \Upsilon(A_1, A_2) = f_0((R/Q)),$$

where  $f$  is a formal real noncommutative power series in the 8 terms  $(R/Q)_j^{(\iota_1, \iota_2)}$ .

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Similarly, a vectorial FQ operation with property (vL) can be written as

$$(2) \quad \Psi(A_1, A_2) = (f_1((R/Q))Q_1, f_2((R/Q))Q_2);$$

and a pseudoscalar FQ operation with property (psL) can be written as

$$(3) \quad \Phi(A_1, A_2) = f_{12}((R/Q))Q_1Q_2.$$

In order to arrive to natural FQ operations, one needs one more assumption, naturality, which is equivalent to uniform analyticity (as opposed to the pointed expansions above), which is also equivalent to conjugation invariance. The reader is advised to look up the discussion in [4]; although we will also address this question later.

Nevertheless, in this paper, our starting point will be simply considering expansions as in (1), (2), (3). Base-point invariance will be required ultimately, but our basic objects will be the pointed expansions as above. So, strictly speaking, we should always use the notation  $\Xi_{(Q_1, Q_2)}(A_1, A_2)$  in order to indicate that the expansion is taken around the Clifford system  $(Q_1, Q_2)$ , but, in general, we will simply write  $\Xi(A_1, A_2)$ . (Later, we will often consider the modification  $\Xi^{\text{ext}}(A_1, A_2) = \Xi_{\mathcal{Q}^{\text{sy}}(A_1, A_2)}(A_1, A_2)$ , which will be our method of choice for obtaining natural operations in this paper.)

**1.2. Convention.** According to this, in what follows, an FQ operation will mean a pointed expansion as above ( $n = 2$ ), either scalar, vectorial, or pseudoscalar; the sign-linear property (sL)/(vL)/(psL) built into the expansion.

**1.3. (a)** In order to deal with the expansions more effectively, we will use the notation

$$\begin{aligned} r_1 &= (R/Q)_1^{(00)}, & r_2 &= (R/Q)_1^{(01)}, & r_3 &= (R/Q)_1^{(10)}, & r_4 &= (R/Q)_1^{(11)}, \\ r_5 &= (R/Q)_2^{(00)}, & r_6 &= (R/Q)_2^{(01)}, & r_7 &= (R/Q)_2^{(10)}, & r_8 &= (R/Q)_2^{(11)}. \end{aligned}$$

This means that the free algebra  $\mathfrak{F}_2$  generated by the Clifford elements  $Q_1, Q_2$  and formal variables  $R_1, R_2$  can also be interpreted as generated by the Clifford elements  $Q_1, Q_2$  and formal variables  $r_1, \dots, r_8$ . In the latter case there are more variables but with (anti)commutation rules with respect to  $Q_1, Q_2$ . We call the infinitesimal base  $r_1, \dots, r_8$  as the split base  $(R/Q)_{\text{split}}$ .

(b) Other choice is the mixed basis  $(R/Q)_{\text{mix}}$  given by  $\hat{r}_1, \dots, \hat{r}_8$ , where

$$(4) \quad \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{r}_6 \\ \hat{r}_7 \\ \hat{r}_8 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{bmatrix}.$$

Here the basis elements are mixed from  $R_1, R_2$  along “characters”:

$$(5) \quad \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{r}_6 \\ \hat{r}_7 \\ \hat{r}_8 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} R_1 Q_1^{-1} \\ Q_1^{-1} R_1 \\ Q_2 R_1 Q_1 Q_2 \\ Q_2 Q_1 R_1 Q_2 \\ R_2 Q_2^{-1} \\ Q_2^{-1} R_2 \\ Q_1 R_2 Q_2 Q_1 \\ Q_1 Q_2 R_2 Q_1 \end{bmatrix}.$$

(The order of elements in the basis is somewhat arbitrary). The commutation rules for this basis are given by

$$\begin{aligned} Q_1 [\hat{r}_1 \ \hat{r}_2 \ \hat{r}_3 \ \hat{r}_4 \ \hat{r}_5 \ \hat{r}_6 \ \hat{r}_7 \ \hat{r}_8] Q_1^{-1} &= [\hat{r}_2 \ \hat{r}_1 \ \hat{r}_3 \ \hat{r}_4 \ -\hat{r}_5 \ -\hat{r}_6 \ -\hat{r}_8 \ -\hat{r}_7], \\ Q_2 [\hat{r}_1 \ \hat{r}_2 \ \hat{r}_3 \ \hat{r}_4 \ \hat{r}_5 \ \hat{r}_6 \ \hat{r}_7 \ \hat{r}_8] Q_2^{-1} &= [-\hat{r}_2 \ -\hat{r}_1 \ \hat{r}_3 \ \hat{r}_4 \ -\hat{r}_5 \ -\hat{r}_6 \ \hat{r}_8 \ \hat{r}_7], \\ Q_1 Q_2 [\hat{r}_1 \ \hat{r}_2 \ \hat{r}_3 \ \hat{r}_4 \ \hat{r}_5 \ \hat{r}_6 \ \hat{r}_7 \ \hat{r}_8] (Q_1 Q_2)^{-1} &= [-\hat{r}_1 \ -\hat{r}_2 \ \hat{r}_3 \ \hat{r}_4 \ \hat{r}_5 \ \hat{r}_6 \ -\hat{r}_7 \ -\hat{r}_8]. \end{aligned}$$

(c) A slight variation is the circular basis  $(R/Q)_{\text{circ}}$  given by  $\tilde{r}_1, \dots, \tilde{r}_8$ , where

$$(6) \quad \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \\ \tilde{r}_4 \\ \tilde{r}_5 \\ \tilde{r}_6 \\ \tilde{r}_7 \\ \tilde{r}_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{r}_6 \\ \hat{r}_7 \\ \hat{r}_8 \end{bmatrix}.$$

The reason for the name will be clear later. The commutation rules for this basis are

$$\begin{aligned} Q_1 [\tilde{r}_1 \ \tilde{r}_2 \ \tilde{r}_3 \ \tilde{r}_4 \ \tilde{r}_5 \ \tilde{r}_6 \ \tilde{r}_7 \ \tilde{r}_8] Q_1^{-1} &= [\tilde{r}_2 \ \tilde{r}_1 \ \tilde{r}_3 \ \tilde{r}_5 \ \tilde{r}_4 \ -\tilde{r}_6 \ -\tilde{r}_8 \ -\tilde{r}_7], \\ Q_2 [\tilde{r}_1 \ \tilde{r}_2 \ \tilde{r}_3 \ \tilde{r}_4 \ \tilde{r}_5 \ \tilde{r}_6 \ \tilde{r}_7 \ \tilde{r}_8] Q_2^{-1} &= [-\tilde{r}_2 \ -\tilde{r}_1 \ \tilde{r}_3 \ \tilde{r}_5 \ \tilde{r}_4 \ -\tilde{r}_6 \ \tilde{r}_8 \ \tilde{r}_7], \\ Q_1 Q_2 [\tilde{r}_1 \ \tilde{r}_2 \ \tilde{r}_3 \ \tilde{r}_4 \ \tilde{r}_5 \ \tilde{r}_6 \ \tilde{r}_7 \ \tilde{r}_8] (Q_1 Q_2)^{-1} &= [-\tilde{r}_1 \ -\tilde{r}_2 \ \tilde{r}_3 \ \tilde{r}_4 \ \tilde{r}_5 \ \tilde{r}_6 \ -\tilde{r}_7 \ -\tilde{r}_8]. \end{aligned}$$

1.4. If  $f$  is one of  $f_0, f_1, f_2, f_{12}$ , then we use the notation

$$f_s((R/Q)) \equiv f_s(r_1, \dots, r_8) = \sum_{r=0}^{\infty} \sum_{k_1, \dots, k_r \in \{1, \dots, 8\}} p_{k_1, \dots, k_r}^{[s]} r_{k_1} \dots r_{k_r}.$$

I. e., when we take noncommutative power series in  $(R/Q)$ , then (somewhat loosely) it will be understood as a noncommutative power series in the (order of the) variables of  $(R/Q)_{\text{split}}$ . These expressions can be realized by other power series  $\hat{f}_s, \tilde{f}_s$  in the mixed and circular bases. So, then  $f((R/Q)_{\text{split}}) = \hat{f}((R/Q)_{\text{mix}}) = \tilde{f}((R/Q)_{\text{circ}})$ . The coefficients of  $f, \hat{f}, \tilde{f}$  can be expressed from each other, using (the inverses of the) matrices from (4) and (6). In particular,

$$\tilde{p}_4^{[s]} = \frac{1}{2}(\hat{p}_4^{[s]} + \hat{p}_5^{[s]}) \quad \text{and} \quad \tilde{p}_5^{[s]} = \frac{1}{2}(\hat{p}_4^{[s]} - \hat{p}_5^{[s]}).$$

For practical purposes, we will also use the notation  $(-1)^{[s]}$ , where  $(-1)^{[0]} = (-1)^{[1]} = (-1)^{[12]} = 1$  and  $(-1)^{[2]} = -1$ .

1.5. **Example.** (a) The expansion of the constant 1 operation (as a scalar operation), in the split basis, is given by

$$\mathbf{P}_0^{[1]} = [1],$$

and all other coefficients, in expansion orders  $r \geq 1$ , are 0. The same applies in the mixed and circular bases, too.

(b) The expansion of the identity operation Id (as a vectorial operation), in the mixed basis, is given by

$$\begin{aligned} \hat{\mathbf{P}}_0^{[1]} &= [1], & \hat{\mathbf{P}}_1^{[1]} &= [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1], \\ \hat{\mathbf{P}}_0^{[2]} &= [1], & \hat{\mathbf{P}}_1^{[2]} &= [-1 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \ -1], \end{aligned}$$

and all other coefficients, in expansion orders  $r \geq 2$ , are 0. In the circular basis it is given by

$$\begin{aligned}\tilde{\mathbf{P}}_0^{[1]} &= [1], & \tilde{\mathbf{P}}_1^{[1]} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \\ \tilde{\mathbf{P}}_0^{[2]} &= [1], & \tilde{\mathbf{P}}_1^{[2]} &= \begin{bmatrix} -1 & 1 & 1 & -1 & 0 & 1 & 1 & -1 \end{bmatrix},\end{aligned}$$

and all other coefficients, in expansion orders  $r \geq 2$ , are 0.

(c) The expansion of the pseudodeterminant operation  $\mathcal{D}(A_1, A_2) = \frac{1}{2}[A_1, A_2]$  (as a pseudoscalar operation), in the mixed basis, is given by

$$\begin{aligned}\hat{\mathbf{P}}_0^{[12]} &= [1], & \hat{\mathbf{P}}_1^{[12]} &= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \\ \hat{\mathbf{P}}_2^{[12]} &= \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \end{bmatrix};\end{aligned}$$

the other coefficients, in expansion orders  $r \geq 3$ , are 0.

1.6. Consider the (conform-)orthogonalization procedures  $\mathcal{O}^{\text{GS}}$ ,  $\mathcal{O}^{\text{Sy}}$ ,  $\mathcal{O}^{\text{fGS}}$ ,  $\mathcal{O}^{\text{fSy}}$  from [4]. Then one can easily see the following:

$(Q_1, Q_2)$  is the Gram-Schmidt orthogonalization of  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$  if and only if  $r_3 = r_4 = r_6 = 0$  in the split base.

$(Q_1, Q_2)$  is the symmetric orthogonalization of  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$  if and only if  $\hat{r}_6 = \hat{r}_7 = \hat{r}_8 = 0$  in the mixed base; and the same applies to the circular base.

$(Q_1, Q_2)$  is the Gram-Schmidt conform-orthogonalization of  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$  if and only if  $r_1 = r_2 = r_3 = r_4 = r_6 = r_8 = 0$  in the split base.

$(Q_1, Q_2)$  is the symmetric conform-orthogonalization of  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$  if and only if  $\hat{r}_1 = \hat{r}_2 = \hat{r}_3 = \hat{r}_6 = \hat{r}_7 = \hat{r}_8 = 0$  in the mixed base; and the same applies to the circular base.

1.7. The simplest invariance property for FQ operations is

(sC) / (vC) / (psC) Clifford conservativity: It means  $\Xi(Q_1, Q_2) = 1 / \Xi(Q_1, Q_2) = (Q_1, Q_2) / \Xi(Q_1, Q_2) = Q_1 Q_2$  respectively.

In terms of the expansions, it can be expressed by  $p^{[0]} = 1 / p^{[1]} = p^{[2]} = 1 / p^{[12]} = 1$  respectively (and the same in other bases), i. e., the leading coefficients of the power series expansion are 1. Together with naturality / conjugation invariance, this implies that  $\Xi$  yields the expected simple results on any Clifford system. We will often (but not always) assume this property.

## 2. PRINCIPAL INVARIANCE PROPERTIES

Here we study the most basic invariance properties. The discussion might seem to be a bit redundant regarding the use of various bases, but it is useful to see how these bases are different from each other.

**2.1. Conjugation invariance (Nat).** This is equivalent to naturality, and base Clifford system invariance. I. e. in the expansions (1), (2), (3) one can use Gram-Schmidt orthogonalization or the symmetric orthogonalization as base system.

In terms of the split basis and the Gram-Schmidt orthogonalization, this means that in the expansions (1), (2), (3), the formal power series  $f(r_1, \dots, r_8)$  can be reconstructed

uniquely from  $f(r_1, r_2, 0, 0, r_5, 0, r_7, r_8)$ , and that any formal power series  $g(r_1, r_2, r_5, r_7, r_8)$  can be prescribed for the latter. Indeed, if  $A = Q + R$ ,  $Q^{\text{GS}} = \underline{Q}^{\text{GS}}(A)$  and  $A = Q^{\text{GS}} + R^{\text{GS}}$ ,

$$(R/Q)_{\text{split}} = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \end{bmatrix},$$

then one can see that

$$(7) \quad (R^{\text{GS}}/Q^{\text{GS}})_{\text{split}} = \begin{bmatrix} r_1 + O(R/Q)^2 & r_2 + O(R/Q)^2 & 0 & \dots \\ \dots & 0 & r_5 + O(R/Q)^2 & 0 & r_7 + O(R/Q)^2 & r_8 - r_4 + O(R/Q)^2 \end{bmatrix}.$$

Hence, when (1), (2), (3) are expanded around  $Q^{\text{GS}}$  (which we can compute) only the 1, 2, 5, 7, 8-coefficients count. This argument is worked out in [4] in the vectorial case, up to order 2.

In terms of the mixed basis and symmetric orthogonalization, this means that in the expansions the formal power series  $f(\hat{r}_1, \dots, \hat{r}_8)$  can be reconstructed from the restrictions  $f(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5, 0, 0, 0)$ , and also that any formal power series  $g(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5)$  can be prescribed for the latter. Using the mixed basis, if  $A = Q + R$ ,  $Q^{\text{Sy}} = \underline{Q}^{\text{Sy}}(A)$  and  $A = Q^{\text{Sy}} + R^{\text{Sy}}$ ,

$$(R/Q)_{\text{mix}} = \begin{bmatrix} \hat{r}_1 & \hat{r}_2 & \hat{r}_3 & \hat{r}_4 & \hat{r}_5 & \hat{r}_6 & \hat{r}_7 & \hat{r}_8 \end{bmatrix},$$

then one can see that

$$(8) \quad (R^{\text{Sy}}/Q^{\text{Sy}})_{\text{mix}} = \begin{bmatrix} \hat{r}_1 + O(R/Q)^{\geq 2} & \hat{r}_2 + O(R/Q)^{\geq 2} & \hat{r}_3 + O(R/Q)^{\geq 2} & \dots \\ \dots & \hat{r}_4 + O(R/Q)^{\geq 2} & \hat{r}_5 + O(R/Q)^{\geq 2} & 0 & 0 & 0 \end{bmatrix}.$$

Hence, when (1), (2), (3) are expanded around  $Q^{\text{Sy}}$  (which we can compute) only the 1, 2, 3, 4, 5-coefficients count. The very same argument applies to the circular basis.

This means that for (pseudo)scalar operations we have  $5^r$  coefficients to be chosen freely in the  $r$ th perturbation level (the pure  $\{1, 2, 5, 7, 8\}$ -terms in the split basis, and the pure  $\{1, 2, 3, 4, 5\}$ -terms in the mixed/circular bases), and  $2 \times 5^r$  coefficients in the vectorial case; and all the other coefficients can be expressed from them explicitly. The actual reductions/extensions are to be found from

$$\Xi_{(Q_1, Q_2)}(Q_1 + R_1, Q_2 + R_2) = \Xi_{\underline{Q}^{\omega}(Q_1 + R_1, Q_2 + R_2)}(Q_1 + R_1, Q_2 + R_2),$$

but they are not particularly simple. (Although, later, we will show explicit recursion relations for them by direct methods.)

There is a small advantage using the mixed/circular bases to the split basis. This is as follows: When we consider an expansion according to (7), and we eliminate the  $\{3, 4, 6\}$ -indices, we see that in the result the coefficient of  $r_4$  gets a contribution from the coefficient of  $r_8$ . This is reflected in the natural coefficient rule  $p_4^{[1]} = p^{[1]} - p_8^{[1]}$  (in the split basis), cf. [4]. On the other hand, in the mixed/circular bases the coefficients of the eliminated indices  $\{6, 7, 8\}$  do not receive contribution from the same order; hence, during the reduction process the coefficient indices  $\{6, 7, 8\}$  “decay” to a lower order:

**2.2. Example.** Similarly, to the example in [4], let  $\Psi$  be a formal FQ operation with (vL),  $n = 2$ ; but now we use the circular base. Then

$$(F_2) \quad \Psi(Q + R)_s = \left( \tilde{p}^{[s]} + \sum_{i=1}^8 \tilde{p}_i^{[s]} \tilde{r}_i + \sum_{i,j=1}^8 \tilde{p}_{ij}^{[s]} \tilde{r}_i \tilde{r}_j + O((R/Q)^{\geq 3}) \right) Q_s, \quad s \in \{1, 2\}.$$

We collect (some of) the coefficients into the scalar matrices  $\mathbf{P}_0^{[s]} = [\tilde{p}^{[s]}]$ , the row matrices  $\tilde{\mathbf{P}}_1^{[s]} = [\tilde{p}_i^{[s]}]_{i=1}^8$ , and the square matrices  $\tilde{\mathbf{P}}_2^{[s]} = [\tilde{p}_{ij}^{[s]}]_{i,j=1}^8$ .

If  $\Psi = \underline{\mathcal{Q}}^{\text{Sy}}$ , then one can check that  $\tilde{\mathbf{P}}_0^{[1]} = [1]$ ,  $\tilde{\mathbf{P}}_0^{[2]} = [1]$ , and

$$\tilde{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{P}}_1^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix},$$

$$\tilde{\mathbf{P}}_2^{[1]} = \frac{1}{2} \begin{bmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & 0 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & -1 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & -1 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 0 & 0 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 1 & 2 \\ -1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{P}}_2^{[2]} = \frac{1}{2} \begin{bmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 1 & 0 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & -1 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & -1 & 1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 0 & 0 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 & 0 & 1 & 1 & -2 \\ -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

Notice that all the boxed entries vanish. Indeed, as the expansion of the symmetric orthogonalization is trivial in itself, the pure  $\{1, 2, 3, 4, 5\}$ -terms must vanish in expansion degrees  $r \geq 1$ .

In case  $\Psi$  is arbitrary, we can compute  $(F_2)$  using  $\tilde{Q} = \underline{\mathcal{Q}}^{\text{Sy}}(A)$  as the base Clifford system, we obtain  $\tilde{\mathbf{P}}_0^{[1]} = [\tilde{p}^{[1]}]$ ,  $\tilde{\mathbf{P}}_0^{[2]} = [\tilde{p}^{[2]}]$ , and

$$\tilde{\mathbf{P}}_1^{[1]} = \begin{bmatrix} \boxed{\tilde{p}_1^{[1]}} & \boxed{\tilde{p}_2^{[1]}} & \boxed{\tilde{p}_3^{[1]}} & \boxed{\tilde{p}_4^{[1]}} & \boxed{\tilde{p}_5^{[1]}} & \tilde{p}^{[1]} & \tilde{p}^{[1]} & \tilde{p}^{[1]} \end{bmatrix},$$

$$\tilde{\mathbf{P}}_1^{[2]} = \begin{bmatrix} \boxed{\tilde{p}_1^{[2]}} & \boxed{\tilde{p}_2^{[2]}} & \boxed{\tilde{p}_3^{[2]}} & \boxed{\tilde{p}_4^{[2]}} & \boxed{\tilde{p}_5^{[2]}} & \tilde{p}^{[2]} & \tilde{p}^{[2]} & -\tilde{p}^{[2]} \end{bmatrix},$$

$$\tilde{\mathbf{P}}_2^{[1]} = \begin{bmatrix} \boxed{\tilde{p}_{1,1}^{[1]}} & \boxed{\tilde{p}_{1,2}^{[1]}} & \boxed{\tilde{p}_{1,3}^{[1]}} \\ \boxed{\tilde{p}_{2,1}^{[1]}} & \boxed{\tilde{p}_{2,2}^{[1]}} & \boxed{\tilde{p}_{2,3}^{[1]}} \\ \boxed{\tilde{p}_{3,1}^{[1]}} & \boxed{\tilde{p}_{3,2}^{[1]}} & \boxed{\tilde{p}_{3,3}^{[1]}} \\ \boxed{\tilde{p}_{4,1}^{[1]}} & \boxed{\tilde{p}_{4,2}^{[1]}} & \boxed{\tilde{p}_{4,3}^{[1]}} \\ \boxed{\tilde{p}_{5,1}^{[1]}} & \boxed{\tilde{p}_{5,2}^{[1]}} & \boxed{\tilde{p}_{5,3}^{[1]}} \\ -\frac{1}{2}\tilde{p}^{[1]} + \frac{1}{2}\tilde{p}_1^{[1]} & -\frac{1}{2}\tilde{p}^{[1]} + \frac{1}{2}\tilde{p}_2^{[1]} & -\frac{1}{2}\tilde{p}^{[1]} + \frac{1}{2}\tilde{p}_3^{[1]} \\ \tilde{p}_1^{[1]} - \tilde{p}_4^{[1]} & -\frac{1}{2}\tilde{p}^{[1]} + \tilde{p}_2^{[1]} - \frac{1}{2}\tilde{p}_3^{[1]} & -\frac{1}{2}\tilde{p}^{[1]} - \frac{1}{2}\tilde{p}_2^{[1]} + \tilde{p}_3^{[1]} \\ -\frac{1}{2}\tilde{p}^{[1]} + \frac{1}{2}\tilde{p}_3^{[1]} & \tilde{p}_5^{[1]} & -\frac{1}{2}\tilde{p}^{[1]} + \frac{1}{2}\tilde{p}_1^{[1]} \end{bmatrix} \dots$$

$$\begin{aligned}
& \dots \left[ \begin{array}{cc|cc|c} \boxed{\tilde{p}_{1,4}^{[1]}} & \boxed{\tilde{p}_{1,5}^{[1]}} & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_1^{[1]} & \tilde{p}_5^{[1]} & -\frac{1}{2}\tilde{p}^{[1]}+\tilde{p}_1^{[1]}-\frac{1}{2}\tilde{p}_3^{[1]} \\ \boxed{\tilde{p}_{2,4}^{[1]}} & \boxed{\tilde{p}_{2,5}^{[1]}} & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_2^{[1]} & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_3^{[1]} & \tilde{p}_2^{[1]}-\tilde{p}_4^{[1]} \\ \boxed{\tilde{p}_{3,4}^{[1]}} & \boxed{\tilde{p}_{3,5}^{[1]}} & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_3^{[1]} & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_2^{[1]} & -\frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_1^{[1]}+\tilde{p}_3^{[1]} \\ \boxed{\tilde{p}_{4,4}^{[1]}} & \boxed{\tilde{p}_{4,5}^{[1]}} & 0 & 0 & -\frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_2^{[1]}+\tilde{p}_4^{[1]} \\ \boxed{\tilde{p}_{5,4}^{[1]}} & \boxed{\tilde{p}_{5,5}^{[1]}} & \tilde{p}_5^{[1]} & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_1^{[1]} & \tilde{p}_5^{[1]} \\ 0 & \tilde{p}_5^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_3^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_2^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_1^{[1]} \\ -\frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_1^{[1]}+\tilde{p}_4^{[1]} & \tilde{p}_5^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_2^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_3^{[1]} & \tilde{p}^{[1]}-\tilde{p}_4^{[1]} \\ 0 & -\frac{1}{2}\tilde{p}^{[1]}+\frac{1}{2}\tilde{p}_2^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_1^{[1]} & -\tilde{p}_5^{[1]} & \frac{1}{2}\tilde{p}^{[1]}-\frac{1}{2}\tilde{p}_3^{[1]} \end{array} \right], \\
& \tilde{\mathbf{P}}_2^{[2]} = \left[ \begin{array}{ccc|c} \boxed{\tilde{p}_{1,1}^{[2]}} & \boxed{\tilde{p}_{1,2}^{[2]}} & \boxed{\tilde{p}_{1,3}^{[2]}} & \\ \boxed{\tilde{p}_{2,1}^{[2]}} & \boxed{\tilde{p}_{2,2}^{[2]}} & \boxed{\tilde{p}_{2,3}^{[2]}} & \\ \boxed{\tilde{p}_{3,1}^{[2]}} & \boxed{\tilde{p}_{3,2}^{[2]}} & \boxed{\tilde{p}_{3,3}^{[2]}} & \\ \boxed{\tilde{p}_{4,1}^{[2]}} & \boxed{\tilde{p}_{4,2}^{[2]}} & \boxed{\tilde{p}_{4,3}^{[2]}} & \dots \\ \boxed{\tilde{p}_{5,1}^{[2]}} & \boxed{\tilde{p}_{5,2}^{[2]}} & \boxed{\tilde{p}_{5,3}^{[2]}} & \\ \frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_1^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_2^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_3^{[2]} & \\ \tilde{p}_1^{[2]}-\tilde{p}_4^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}+\tilde{p}_2^{[2]}-\frac{1}{2}\tilde{p}_3^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_2^{[2]}+\tilde{p}_3^{[2]} & \\ -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_3^{[2]} & \tilde{p}_5^{[2]} & \frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_1^{[2]} & \end{array} \right] \\
& \dots \left[ \begin{array}{cc|cc|c} \boxed{\tilde{p}_{1,4}^{[2]}} & \boxed{\tilde{p}_{1,5}^{[2]}} & \frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_1^{[2]} & \tilde{p}_5^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}-\tilde{p}_1^{[2]}-\frac{1}{2}\tilde{p}_3^{[2]} \\ \boxed{\tilde{p}_{2,4}^{[2]}} & \boxed{\tilde{p}_{2,5}^{[2]}} & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_2^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_3^{[2]} & -\tilde{p}_2^{[2]}-\tilde{p}_4^{[2]} \\ \boxed{\tilde{p}_{3,4}^{[2]}} & \boxed{\tilde{p}_{3,5}^{[2]}} & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_3^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_2^{[2]} & \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_1^{[2]}-\tilde{p}_3^{[2]} \\ \boxed{\tilde{p}_{4,4}^{[2]}} & \boxed{\tilde{p}_{4,5}^{[2]}} & 0 & 0 & -\frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_2^{[2]}-\tilde{p}_4^{[2]} \\ \boxed{\tilde{p}_{5,4}^{[2]}} & \boxed{\tilde{p}_{5,5}^{[2]}} & \tilde{p}_5^{[2]} & \frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_1^{[2]} & -\tilde{p}_5^{[2]} \\ 0 & \tilde{p}_5^{[2]} & \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_3^{[2]} & \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_2^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_1^{[2]} \\ \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_1^{[2]}+\tilde{p}_4^{[2]} & \tilde{p}_5^{[2]} & \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_2^{[2]} & \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_3^{[2]} & -\tilde{p}^{[2]}-\tilde{p}_4^{[2]} \\ 0 & -\frac{1}{2}\tilde{p}^{[2]}+\frac{1}{2}\tilde{p}_2^{[2]} & -\frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_1^{[2]} & -\tilde{p}_5^{[2]} & \frac{1}{2}\tilde{p}^{[2]}-\frac{1}{2}\tilde{p}_3^{[2]} \end{array} \right].
\end{aligned}$$

Hence, we can eliminate the coefficients which are not in the boxed positions. One can see that the coefficients with any  $\{6, 7, 8\}$ -indices decay to linear combinations of coefficients with only  $\{1, 2, 3, 4, 5\}$ -indices but of lower number.

**2.3. Example.** Similarly, in the mixed basis, we find that naturality implies

$$\begin{aligned}
\hat{\mathbf{P}}_0^{[0]} &= [\hat{p}^{[0]}], \quad \hat{\mathbf{P}}_1^{[0]} = \left[ \begin{array}{ccccc} \boxed{\hat{p}_1^{[0]}} & \boxed{\hat{p}_2^{[0]}} & \boxed{\hat{p}_3^{[0]}} & \boxed{\hat{p}_4^{[0]}} & \boxed{\hat{p}_5^{[0]}} & 0 & 0 & 0 \end{array} \right], \\
\hat{\mathbf{P}}_0^{[1]} &= [\hat{p}^{[1]}], \quad \hat{\mathbf{P}}_1^{[1]} = \left[ \begin{array}{ccccc} \boxed{\hat{p}_1^{[1]}} & \boxed{\hat{p}_2^{[1]}} & \boxed{\hat{p}_3^{[1]}} & \boxed{\hat{p}_4^{[1]}} & \boxed{\hat{p}_5^{[1]}} & \hat{p}^{[1]} & \hat{p}^{[1]} & \hat{p}^{[1]} \end{array} \right],
\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{P}}_0^{[2]} &= [\hat{p}^{[2]}], & \hat{\mathbf{P}}_1^{[2]} &= \begin{bmatrix} \boxed{\hat{p}_1^{[2]}} & \boxed{\hat{p}_2^{[2]}} & \boxed{\hat{p}_3^{[2]}} & \boxed{\hat{p}_4^{[2]}} & \boxed{\hat{p}_5^{[2]}} & \hat{p}^{[2]} & \hat{p}^{[2]} & -\hat{p}^{[2]} \end{bmatrix}, \\ \hat{\mathbf{P}}_0^{[12]} &= [\hat{p}^{[12]}], & \hat{\mathbf{P}}_1^{[12]} &= \begin{bmatrix} \boxed{\hat{p}_1^{[12]}} & \boxed{\hat{p}_2^{[12]}} & \boxed{\hat{p}_3^{[12]}} & \boxed{\hat{p}_4^{[12]}} & \boxed{\hat{p}_5^{[12]}} & 0 & 2\hat{p}^{[12]} & 0 \end{bmatrix}.\end{aligned}$$

**2.4. Transposition invariance (Opp).** For any algebra  $\mathfrak{A}$ , there is an opposite algebra  $\mathfrak{A}^{\text{opp}}$ . Using the notation

$$(B_1, B_2)^{\text{opp}} = (B_1^{\text{opp}}, B_2^{\text{opp}}),$$

transposition invariance for scalar and vectorial operations can be expressed as

$$\Xi_{(Q_1, Q_2)}(A_1, A_2)^{\text{opp}} = \Xi_{(Q_1^{\text{opp}}, Q_2^{\text{opp}})}(A_1^{\text{opp}}, A_2^{\text{opp}});$$

and for pseudoscalar operations,

$$\Xi_{(Q_1, Q_2)}(A_1, A_2)^{\text{opp}} = -\Xi_{(Q_1^{\text{opp}}, Q_2^{\text{opp}})}(A_1^{\text{opp}}, A_2^{\text{opp}}).$$

This invariance means that the FQ operation does not favor right or left.

Regarding the expansions (1), (2), (3), transposition invariance means that in the splitting basis

$$\begin{aligned}f_0(r_1, \dots, r_8) &= f_0^{\text{opp}}(r_1, r_2, -r_3, -r_4, r_5, -r_6, r_7, -r_8), \\ f_1(r_1, \dots, r_8) &= f_1^{\text{opp}}(r_1, r_2, r_3, r_4, r_5, -r_6, -r_7, r_8), \\ f_2(r_1, \dots, r_8) &= f_2^{\text{opp}}(r_1, -r_2, -r_3, r_4, r_5, r_6, r_7, r_8), \\ f_{12}(r_1, \dots, r_8) &= f_{12}^{\text{opp}}(r_1, -r_2, r_3, -r_4, r_5, r_6, -r_7, -r_8);\end{aligned}$$

in the mixed basis

$$\begin{aligned}\hat{f}_0(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_0^{\text{opp}}(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, -\hat{r}_5, -\hat{r}_6, -\hat{r}_7, -\hat{r}_8), \\ \hat{f}_1(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_1^{\text{opp}}(\hat{r}_2, \hat{r}_1, \hat{r}_3, \hat{r}_4, \hat{r}_5, \hat{r}_6, \hat{r}_8, \hat{r}_7), \\ \hat{f}_2(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_2^{\text{opp}}(-\hat{r}_2, -\hat{r}_1, \hat{r}_3, \hat{r}_4, \hat{r}_5, \hat{r}_6, -\hat{r}_8, -\hat{r}_7), \\ \hat{f}_{12}(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_{12}^{\text{opp}}(-\hat{r}_1, -\hat{r}_2, \hat{r}_3, \hat{r}_4, -\hat{r}_5, -\hat{r}_6, \hat{r}_7, \hat{r}_8);\end{aligned}$$

in the circular basis

$$\begin{aligned}\tilde{f}_0(\tilde{r}_1, \dots, \tilde{r}_8) &= \tilde{f}_0^{\text{opp}}(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_5, \tilde{r}_4, -\tilde{r}_6, -\tilde{r}_7, -\tilde{r}_8), \\ \tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8) &= \tilde{f}_1^{\text{opp}}(\tilde{r}_2, \tilde{r}_1, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6, \tilde{r}_8, \tilde{r}_7), \\ \tilde{f}_2(\tilde{r}_1, \dots, \tilde{r}_8) &= \tilde{f}_2^{\text{opp}}(-\tilde{r}_2, -\tilde{r}_1, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6, -\tilde{r}_8, -\tilde{r}_7), \\ \tilde{f}_{12}(\tilde{r}_1, \dots, \tilde{r}_8) &= \tilde{f}_{12}^{\text{opp}}(-\tilde{r}_1, -\tilde{r}_2, \tilde{r}_3, \tilde{r}_5, \tilde{r}_4, -\tilde{r}_6, \tilde{r}_7, \tilde{r}_8).\end{aligned}$$

(Here  $f^{\text{opp}}$  means that the order of the products is reversed.) In fact, if  $\Xi$  is an FQ operation, then we can define  $\Xi^{\text{opp}}$  by

$$\Xi^{\text{opp}}(A_1, A_2) = \pm \Xi(A_1^{\text{opp}}, A_2^{\text{opp}})^{\text{opp}}$$

(minus sign in the pseudoscalar case). Now, transposition invariance means  $\Xi = \Xi^{\text{opp}}$ ; and the RHS of the equations above inform us about the expansion terms of  $\Xi^{\text{opp}}$ .

In particular, in the mixed base, transposition invariance implies

$$\begin{aligned}\begin{bmatrix} \hat{p}_1^{[0]} & \hat{p}_2^{[0]} & \hat{p}_3^{[0]} & \hat{p}_4^{[0]} & \hat{p}_5^{[0]} \end{bmatrix} &= \begin{bmatrix} \hat{p}_1^{[0]} & \hat{p}_2^{[0]} & \hat{p}_3^{[0]} & \hat{p}_4^{[0]} & 0 \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[1]} & \hat{p}_2^{[1]} & \hat{p}_3^{[1]} & \hat{p}_4^{[1]} & \hat{p}_5^{[1]} \end{bmatrix} &= \begin{bmatrix} \hat{p}_1^{[1]} & \hat{p}_1^{[1]} & \hat{p}_3^{[1]} & \hat{p}_4^{[1]} & \hat{p}_5^{[1]} \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[2]} & \hat{p}_2^{[2]} & \hat{p}_3^{[2]} & \hat{p}_4^{[2]} & \hat{p}_5^{[2]} \end{bmatrix} &= \begin{bmatrix} \hat{p}_1^{[2]} & -\hat{p}_1^{[2]} & \hat{p}_3^{[2]} & \hat{p}_4^{[2]} & \hat{p}_5^{[2]} \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[12]} & \hat{p}_2^{[12]} & \hat{p}_3^{[12]} & \hat{p}_4^{[12]} & \hat{p}_5^{[12]} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \hat{p}_3^{[12]} & \hat{p}_4^{[12]} & 0 \end{bmatrix}.\end{aligned}$$



**2.5. Symmetry ( $\Sigma_2$ ).** Using the notation

$$(B_1, B_2)^{\leftrightarrow} = (B_2, B_1),$$

rotation invariance for scalar operations can be expressed as

$$\Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_2, Q_1)}(A_2, A_1),$$

for vectorial operations

$$\Xi_{(Q_1, Q_2)}(A_1, A_2)^{\leftrightarrow} = \Xi_{(Q_2, Q_1)}(A_2, A_1),$$

and for pseudoscalar operations

$$\Xi_{(Q_1, Q_2)}(A_1, A_2) = -\Xi_{(Q_2, Q_1)}(A_2, A_1).$$

Regarding the expansions (1), (2), (3), symmetry means that in the splitting basis

$$\begin{aligned} f_0(r_1, \dots, r_8) &= f_0(r_5, r_7, r_6, r_8, r_1, r_3, r_2, r_4), \\ f_2(r_1, \dots, r_8) &= f_1(r_5, r_7, r_6, r_8, r_1, r_3, r_2, r_4), \\ f_{12}(r_1, \dots, r_8) &= f_{12}(r_5, r_7, r_6, r_8, r_1, r_3, r_2, r_4); \end{aligned}$$

in the mixed basis

$$\begin{aligned} \hat{f}_0(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_0(-\hat{r}_1, \hat{r}_2, \hat{r}_3, -\hat{r}_4, -\hat{r}_5, \hat{r}_6, \hat{r}_7, -\hat{r}_8), \\ \hat{f}_2(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_1(-\hat{r}_1, \hat{r}_2, \hat{r}_3, -\hat{r}_4, -\hat{r}_5, \hat{r}_6, \hat{r}_7, -\hat{r}_8), \\ \hat{f}_{12}(\hat{r}_1, \dots, \hat{r}_8) &= \hat{f}_{12}(-\hat{r}_1, \hat{r}_2, \hat{r}_3, -\hat{r}_4, -\hat{r}_5, \hat{r}_6, \hat{r}_7, -\hat{r}_8); \end{aligned}$$

and the same scheme works in the circular basis. If we use the notation

$$(\pi^\pm g)(\hat{r}_1, \dots, \hat{r}_8) = \frac{1}{2}(g(\hat{r}_1, \dots, \hat{r}_8) \pm g(-\hat{r}_1, \hat{r}_2, \hat{r}_3, -\hat{r}_4, -\hat{r}_5, \hat{r}_6, \hat{r}_7, -\hat{r}_8)),$$

then symmetry can be written as

$$\pi^- \hat{f}_0 = 0, \quad \pi^+ \hat{f}_2 = \pi^+ \hat{f}_1, \quad \pi^- \hat{f}_2 = -\pi^- \hat{f}_1, \quad \pi^- \hat{f}_{12} = 0,$$

respectively; and the same scheme works in the circular basis.

In particular, in the mixed base, symmetry implies

$$\begin{aligned} \begin{bmatrix} \hat{p}_1^{[0]} & \hat{p}_2^{[0]} & \hat{p}_3^{[0]} & \hat{p}_4^{[0]} & \hat{p}_5^{[0]} \end{bmatrix} &= \begin{bmatrix} 0 & \hat{p}_2^{[0]} & \hat{p}_3^{[0]} & 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[2]} & \hat{p}_2^{[2]} & \hat{p}_3^{[2]} & \hat{p}_4^{[2]} & \hat{p}_5^{[2]} \end{bmatrix} &= \begin{bmatrix} -\hat{p}_1^{[1]} & \hat{p}_2^{[1]} & \hat{p}_3^{[1]} & -\hat{p}_4^{[1]} & -\hat{p}_5^{[1]} \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[12]} & \hat{p}_2^{[12]} & \hat{p}_3^{[12]} & \hat{p}_4^{[12]} & \hat{p}_5^{[12]} \end{bmatrix} &= \begin{bmatrix} 0 & \hat{p}_2^{[12]} & \hat{p}_3^{[12]} & 0 & 0 \end{bmatrix}. \end{aligned}$$

**2.6. Orthogonal invariance ( $\mathcal{O}_2$ ).** Using the notation

$$\text{rot}_\phi(B_1, B_2) = (B_1 \cos \phi + B_2 \sin \phi, B_2 \cos \phi - B_1 \sin \phi),$$

this invariance for (pseudo)scalar operations can be expressed as

$$\Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{\text{rot}_\phi(Q_1, Q_2)}(\text{rot}_\phi(A_1, A_2)),$$

and for vectorial operations

$$\text{rot}_\phi(\Xi_{(Q_1, Q_2)}(A_1, A_2)) = \Xi_{\text{rot}_\phi(Q_1, Q_2)}(\text{rot}_\phi(A_1, A_2)).$$

(Strictly speaking, this is rotational invariance, but due to (sL) / (vL) / (psL), this is equivalent to orthogonal invariance, and, in particular, it is stronger than symmetry.)

In order to describe this invariance, it is better pass to a formal commutative variable  $t$  with  $t^2 = 0$ , instead of  $\phi$ . Then

$$\text{rot}_t(B_1, B_2) = (B_1 + B_2 t, B_2 - B_1 t).$$

Coefficients of  $t$ , in calculations like above, correspond to derivations, or vector fields. According to this, using  $t$  is equivalent to using  $\phi$ . One can also see that considering several successive such rotations by  $t_1, t_2, \dots$  is equivalent a rotation by  $t = t_1 + t_2 + \dots$ , where the restriction  $t^2 = 0$  is dropped.

Orthogonal invariance is best to be described using the circular basis. The action  $\text{rot}_t$  induces a derivation  $\Delta$  on  $\mathfrak{F}_2$  given by

$$\begin{aligned}\Delta(Q_1) &= Q_2 = -Q_1 \cdot Q_1 Q_2, & \Delta(Q_2) &= -Q_1 = -Q_2 \cdot Q_1 Q_2, \\ \Delta(\tilde{r}_1) &= -2\tilde{r}_1 Q_1 Q_2, & \Delta(\tilde{r}_2) &= 0, & \Delta(\tilde{r}_3) &= 0, & \Delta(\tilde{r}_4) &= -2\tilde{r}_4 Q_1 Q_2, \\ \Delta(\tilde{r}_5) &= 2\tilde{r}_5 Q_1 Q_2, & \Delta(\tilde{r}_6) &= 0, & \Delta(\tilde{r}_7) &= 0, & \Delta(\tilde{r}_8) &= -2\tilde{r}_8 Q_1 Q_2.\end{aligned}$$

Then, in terms of the expansions (1), (2), (3), orthogonal invariance is equivalent to

$$\begin{aligned}\Delta(\tilde{f}_0(\tilde{r}_1, \dots, \tilde{r}_8)) &= 0, \\ \Delta(\tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8)Q_1) &= \tilde{f}_2(\tilde{r}_1, \dots, \tilde{r}_8)Q_2, \\ \Delta(\tilde{f}_2(\tilde{r}_1, \dots, \tilde{r}_8)Q_2) &= -\tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8)Q_1, \\ \Delta(\tilde{f}_{12}(\tilde{r}_1, \dots, \tilde{r}_8)Q_1 Q_2) &= 0.\end{aligned}$$

Taking into account that  $\Delta$  commutes with  $\pi^\pm$ , and using the notation  $\Delta_0 = -\frac{\Delta}{2}$ ,

$$\begin{aligned}\Delta_0(\tilde{r}_1) &= \tilde{r}_1 Q_1 Q_2, & \Delta_0(\tilde{r}_2) &= 0, & \Delta_0(\tilde{r}_3) &= 0, & \Delta_0(\tilde{r}_4) &= \tilde{r}_4 Q_1 Q_2, \\ \Delta_0(\tilde{r}_5) &= -\tilde{r}_5 Q_1 Q_2, & \Delta_0(\tilde{r}_6) &= 0, & \Delta_0(\tilde{r}_7) &= 0, & \Delta_0(\tilde{r}_8) &= \tilde{r}_8 Q_1 Q_2;\end{aligned}$$

we see that orthogonal invariance means

$$\begin{aligned}\Delta_0(\tilde{f}_0(\tilde{r}_1, \dots, \tilde{r}_8)) &= 0, \\ \Delta_0(\tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8))(Q_1 Q_2)^{-1} &= \pi^- \tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8), \\ \tilde{f}_2(\tilde{r}_1, \dots, \tilde{r}_8) &= \tilde{f}_1(-\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, -\tilde{r}_4, -\tilde{r}_5, \tilde{r}_6, \tilde{r}_7, -\tilde{r}_8), \\ \Delta_0(\tilde{f}_{12}(\tilde{r}_1, \dots, \tilde{r}_8)) &= 0.\end{aligned}$$

This goes beyond symmetry by

$$\begin{aligned}\Delta_0(\pi^+ \tilde{f}_0(\tilde{r}_1, \dots, \tilde{r}_8)) &= 0, \\ \Delta_0(\pi^+ \tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8)) &= 0, \\ \Delta_0(\pi^- \tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8))(Q_1 Q_2)^{-1} - \pi^- \tilde{f}_1(\tilde{r}_1, \dots, \tilde{r}_8) &= 0, \\ \Delta_0(\pi^+ \tilde{f}_{12}(\tilde{r}_1, \dots, \tilde{r}_8)) &= 0.\end{aligned}$$

The actions on left above act monomially on the expressions of  $f_s(\tilde{r}_1, \dots, \tilde{r}_8)$ . Indeed this follows from the shape of the  $\Delta_0$  and that  $Q_1 Q_2$  (anti)commutes with the  $\tilde{r}_i$ . Consequently, invariance means that certain monomial coefficients in  $\tilde{f}_s(\tilde{r}_1, \dots, \tilde{r}_8)$  vanish. For example, in terms of pseudoscalar invariance, take a monomial  $M$  in  $\tilde{r}_1, \dots, \tilde{r}_8$ . Then  $\Delta_0(M) = \lambda M$ . Now, if  $\lambda \neq 0$  then it means that the coefficient of  $M$  in  $\tilde{f}_{12}$  must vanish; if  $\lambda = 0$  then there is no restriction. Hence, one can check the invariance conditions for monomials (in the circular basis) very fast; nevertheless there are nontrivial patterns. For example:

**2.7. Lemma.** *Consider the coefficients  $\tilde{p}_{i_1, \dots, i_r}^{[s]}$  where  $\{i_1, \dots, i_r\} \subset \{4, 5\}$ . Then the cases when rotation invariance does not imply their vanishing are when*

$$\text{Mult}_{(i_1, \dots, i_r)}^4 - \text{Mult}_{(i_1, \dots, i_r)}^5 = 0 \quad \text{and} \quad [s] \text{ is arbitrary}$$

or

$$\text{Mult}_{(i_1, \dots, i_r)}^4 - \text{Mult}_{(i_1, \dots, i_r)}^5 = 1 \quad \text{and} \quad [s] \in \{[1], [2]\}.$$

(Here  $\text{Mult}_\iota^x$  denotes the multiplicity of  $x$  in the sequence  $\iota$ .)

*Proof.* One can show that, with this restricted choice of indices, for  $\tilde{r}_\iota = \tilde{r}_{i_1} \cdots \tilde{r}_{i_r}$ , the identity  $\Delta_0 \tilde{r}_\iota = (\text{Mult}_\iota^4 - \text{Mult}_\iota^5) \tilde{r}_\iota$  holds. (This is easy to establish if  $\iota$  is composed purely from 4's or 5's; then we can show that inserting 45's or 54's does not change the eigenvalue.) From this, and the previous discussion, one can deduce the statement.  $\square$

**2.8. Lemma.** *For a word  $w$  composed from  $\{1, 2\}$ , let  $\text{red } w$  denote its shortest reduction by the rules  $11 = \lambda$  and  $22 = \lambda$  (where  $\lambda$  is the empty word).*

*Consider the coefficients  $\hat{p}_{i_1, \dots, i_r}^{[s]}$  where  $\{i_1, \dots, i_r\} \subset \{1, 2\}$ . Then the cases when rotation invariance does not imply their vanishing are when*

$$\text{red}(i_1, \dots, i_r) \in \{\lambda, (2)\} \quad \text{and} \quad [s] \text{ is arbitrary}$$

or

$$\text{red}(i_1, \dots, i_r) \in \{(1), (2, 1)\} \quad \text{and} \quad [s] \in \{[1], [2]\}.$$

*Proof.* With this restricted choice of indices, consider  $\tilde{r}_\iota = \tilde{r}_{i_1} \cdots \tilde{r}_{i_r}$ . One can show that if  $\iota$  is reduced (i. e. it contains 1 and 2 alternating), then  $\Delta_0 \tilde{r}_\iota = -(-1)^{i_r} \text{Mult}_\iota^1 \cdot \tilde{r}_\iota$  holds (vanishes for  $\iota = \lambda$ ). Furthermore, one can show that inserting 11's or 22's does not change the eigenvalue. From this, and the previous discussion, one can deduce the statement.  $\square$

In the mixed basis, the differential action is still simple enough,

$$\begin{aligned} \Delta_0(\hat{r}_1) &= \hat{r}_1 Q_1 Q_2, & \Delta_0(\hat{r}_2) &= 0, & \Delta_0(\hat{r}_3) &= 0, & \Delta_0(\hat{r}_4) &= \hat{r}_5 Q_1 Q_2, \\ \Delta_0(\hat{r}_5) &= \hat{r}_4 Q_1 Q_2, & \Delta_0(\hat{r}_6) &= 0, & \Delta_0(\hat{r}_7) &= 0, & \Delta_0(\hat{r}_8) &= \hat{r}_8 Q_1 Q_2; \end{aligned}$$

but it is no longer diagonalized by the monomials, hence the result is more complicated. In particular, in the mixed base, orthogonal invariance implies

$$\begin{aligned} \begin{bmatrix} \hat{p}_1^{[0]} & \hat{p}_2^{[0]} & \hat{p}_3^{[0]} & \hat{p}_4^{[0]} & \hat{p}_5^{[0]} \end{bmatrix} &= \begin{bmatrix} 0 & \hat{p}_2^{[0]} & \hat{p}_3^{[0]} & 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[1]} & \hat{p}_2^{[1]} & \hat{p}_3^{[1]} & \hat{p}_4^{[1]} & \hat{p}_5^{[1]} \end{bmatrix} &= \begin{bmatrix} \hat{p}_1^{[1]} & \hat{p}_2^{[1]} & \hat{p}_3^{[1]} & \hat{p}_4^{[1]} & \hat{p}_4^{[1]} \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[2]} & \hat{p}_2^{[2]} & \hat{p}_3^{[2]} & \hat{p}_4^{[2]} & \hat{p}_5^{[2]} \end{bmatrix} &= \begin{bmatrix} -\hat{p}_1^{[1]} & \hat{p}_2^{[1]} & \hat{p}_3^{[1]} & -\hat{p}_4^{[1]} & -\hat{p}_4^{[1]} \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_1^{[12]} & \hat{p}_2^{[12]} & \hat{p}_3^{[12]} & \hat{p}_4^{[12]} & \hat{p}_5^{[12]} \end{bmatrix} &= \begin{bmatrix} 0 & \hat{p}_2^{[12]} & \hat{p}_3^{[12]} & 0 & 0 \end{bmatrix}. \end{aligned}$$

**2.9.** Looking for nice FQ operations, the invariance properties above are most desirable, although they may be invalid for some auxiliary constructions. Some other, stronger, conditions abstract the basic properties of floating-analytic expansions:

(**Biv'**)  $\Xi$  is bivariant if for  $\theta_1, \theta_2 \approx 1$ , the identity

$$\Xi(\theta_1 A_1 \theta_2, \theta_1 A_2 \theta_2) = \theta_1 \cdot \Xi(A_1, A_2) \cdot \theta_2$$

holds. We talk about symmetric bivariate if this holds with the choice  $\theta_1 = \theta_2$ .

(**Biv''**)  $\Xi$  is antivariant, if for  $\theta_1, \theta_2 \approx 1$ , the identity

$$\Xi(\theta_1 A_1 \theta_2, \theta_1 A_2 \theta_2) = \theta_2^{-1} \cdot \Xi(A_1, A_2) \cdot \theta_1^{-1}$$

holds. We talk about symmetric antivariance if this holds with the choice  $\theta_1 = \theta_2$ .

(**Liv**)  $\Xi$  is left-variant if for  $\theta_1, \theta_2 \approx 1$ , the identity

$$\Xi(\theta_1 A_1 \theta_2, \theta_1 A_2 \theta_2) = \theta_1 \cdot \Xi(A_1, A_2) \cdot \theta_1^{-1}$$

holds. We talk about symmetric left-variance if this holds with the choice  $\theta_1 = \theta_2$ .

(**Riv**)  $\Xi$  is right-variant if for  $\theta_1, \theta_2 \approx 1$ , the identity

$$\Xi(\theta_1 A_1 \theta_2, \theta_1 A_2 \theta_2) = \theta_2^{-1} \cdot \Xi(A_1, A_2) \cdot \theta_2$$

holds. We talk about symmetric right-variance if this holds with the choice  $\theta_1 = \theta_2$ .

### 3. SCALING INVARIANCE PROPERTIES

**3.1.  $\alpha$ -Homogeneity ( $\underline{H}^\alpha$ ).** We formulate this invariance infinitesimally. Here  $t$  is a commutative formal variable with  $t^2 = 0$ . Then,  $\alpha$ -homogeneity can be expressed as

$$(1 + \alpha t) \cdot \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t)A_1, (1 + t)A_2)$$

(in the vectorial case the product is taken componentwise).

Here we will use the mixed basis. Then there is a derivation  $\Delta_3$  on  $(R/Q)$  given by

$$\begin{aligned} \Delta_3(\hat{r}_1) &= \hat{r}_1, & \Delta_3(\hat{r}_2) &= \hat{r}_2, & \Delta_3(\hat{r}_3) &= \hat{r}_3 + 1, & \Delta_3(\hat{r}_4) &= \hat{r}_4, \\ \Delta_3(\hat{r}_5) &= \hat{r}_5, & \Delta_3(\hat{r}_6) &= \hat{r}_6, & \Delta_3(\hat{r}_7) &= \hat{r}_7, & \Delta_3(\hat{r}_8) &= \hat{r}_8 \end{aligned}$$

(which does not respect the filtration, but anyway).

Then,  $\alpha$ -homogeneity corresponds to the collection of relations

$$\hat{p}_{3, k_1, k_2, \dots, k_r}^{[s]} + \hat{p}_{k_1, 3, k_2, \dots, k_r}^{[s]} + \dots + \hat{p}_{k_1, k_2, \dots, k_r, 3}^{[s]} = (\alpha - r)\hat{p}_{k_1, k_2, \dots, k_r}^{[s]}.$$

Ultimately, this means that all the  $\hat{p}_{3, k_1, k_2, \dots, k_r}^{[s]}$  (coefficients starting with index 3) can be eliminated. In particular, in the Clifford conservative case, this implies  $\hat{p}_3^{[s]} = \alpha$ .

**3.2.  $\alpha$ -Equiaffinity ( $\underline{E}^\alpha$ ).** Again, we formulate this condition infinitesimally. Let  $t$  be a commutative formal variable with  $t^2 = 0$ , and let

$$E^\alpha(B_1, B_2) := ((1 + \alpha t)B_1, (1 - \alpha t)B_2).$$

Then, in the vectorial case,  $\alpha$ -equiaffinity can be expressed as

$$E^\alpha \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t)A_1, (1 - t)A_2).$$

In the (pseudo)scalar case, it is defined by

$$(1 + \alpha t) \cdot \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t)A_1, (1 - t)A_2).$$

Here,  $E^\alpha$  induces a derivation  $\Delta_4$  on  $(R/Q)$  given by

$$\begin{aligned} \Delta_4(\hat{r}_1) &= \hat{r}_2, & \Delta_4(\hat{r}_2) &= \hat{r}_1, & \Delta_4(\hat{r}_3) &= \hat{r}_4, & \Delta_4(\hat{r}_4) &= \hat{r}_3 + 1, \\ \Delta_4(\hat{r}_5) &= \hat{r}_6, & \Delta_4(\hat{r}_6) &= \hat{r}_5, & \Delta_4(\hat{r}_7) &= \hat{r}_8, & \Delta_4(\hat{r}_8) &= \hat{r}_7. \end{aligned}$$

Then,  $\alpha$ -equiaffinity corresponds to the collection of relations

$$(S4) \quad \hat{p}_{4, k_1, k_2, \dots, k_r}^{[s]} + \hat{p}_{k_1, 4, k_2, \dots, k_r}^{[s]} + \dots + \hat{p}_{k_1, k_2, \dots, k_r, 4}^{[s]} = (-1^{[s]})\alpha \hat{p}_{k_1, k_2, \dots, k_r}^{[s]} - (\tilde{\Delta}_4 \hat{p})_{k_1, k_2, \dots, k_r}^{[s]};$$

where  $(\tilde{\Delta}_4 \hat{p})_{k_1, k_2, \dots, k_r}^{[s]}$  is an appropriate linear combination of various  $\hat{p}_{l_1, l_2, \dots, l_r}^{[s]}$ . In particular, in the Clifford conservative case, this implies  $\hat{p}_4^{[s]} = (-1^{[s]})\alpha$ .

**3.3.  $\alpha$ -Skew-equiaffinity ( $\underline{CE}^\alpha$ ).** Let  $t$  be a formal variable with  $t^2 = 0$ , but which anticommutes with the  $Q_i$  and  $R_i$  (hence commutes with the  $\hat{r}_i$ ) and let

$$CE^\alpha(B_1, B_2) := ((1 + \alpha t)B_1, (1 - \alpha t)B_2).$$

Then, in the vectorial case,  $\alpha$ -skew-equiaffinity can be expressed as

$$CE^\alpha \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t)A_1, (1 - t)A_2).$$

In the (pseudo)scalar case, it is defined by

$$(1 + \alpha t) \cdot \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t)A_1, (1 - t)A_2).$$

Here,  $CE^\alpha$  induces a derivation  $\Delta_5$  on  $(R/Q)$  given by

$$\begin{aligned} \Delta_5(\hat{r}_1) &= \hat{r}_7, & \Delta_5(\hat{r}_2) &= \hat{r}_8, & \Delta_5(\hat{r}_3) &= \hat{r}_5, & \Delta_5(\hat{r}_4) &= \hat{r}_6, \\ \Delta_5(\hat{r}_5) &= \hat{r}_3 + 1, & \Delta_5(\hat{r}_6) &= \hat{r}_4, & \Delta_5(\hat{r}_7) &= \hat{r}_1, & \Delta_5(\hat{r}_8) &= \hat{r}_2. \end{aligned}$$

Then,  $\alpha$ -skew-equiaffinity corresponds to the collection of relations

$$(S5) \quad \hat{p}_{5,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,5,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,5}^{[s]} = (-1^{[s]})\alpha\hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_5\hat{p})_{k_1,k_2,\dots,k_r}^{[s]};$$

where  $(\tilde{\Delta}_5\hat{p})_{k_1,k_2,\dots,k_r}^{[s]}$  is an appropriate linear combination of various  $\hat{p}_{l_1,l_2,\dots,l_r}^{[s]}$ . In particular, in the Clifford conservative case, this implies  $\hat{p}_5^{[s]} = (-1^{[s]})\alpha$ .

**3.4.  $\alpha$ -Superhomogeneity ( $\underline{SH}^\alpha$ ) and  $\alpha$ -super-equiaffinity ( $\underline{SE}^\alpha$ ).** Let  $F_1$  and  $F_2$  be involutions ( $F_i^2 = 1$ ) of order 0, such that  $F_i$  anticommutes with  $Q_i, R_i$  and commutes with  $Q_{1-i}, R_{1-i}$ , and  $F_1$  commutes with  $F_2$ ; and let  $t$  be an infinitesimal variable (i. e. of order 1). This describes an extension of  $\mathfrak{F}_2$ . Then  $t_1 = F_1t, t_2 = F_2t$  can be considered as formal variables with  $t_1^2 = t_2^2 = t_1t_2 = t_2t_1 = 0$ , such that  $t_i$  anticommutes with the  $Q_i$  and  $R_i$  and commutes with  $Q_{1-i}$  and  $R_{1-i}$ ; hence  $t_i$  commutes with the  $\hat{r}_j$ . Let

$$S^\alpha(B_1, B_2) := ((1 + \alpha t_1)B_1, (1 + \alpha t_2)B_2).$$

$S^\alpha$  induces derivations on  $(R/Q)$ :  $\frac{1}{2}(\Delta_1 + \Delta_2)$  from  $t_1$ , and  $\frac{1}{2}(\Delta_2 - \Delta_1)$  from  $t_2$ , where

$$\begin{aligned} \Delta_2(\hat{r}_1) &= \hat{r}_4, & \Delta_2(\hat{r}_2) &= \hat{r}_3 + 1, & \Delta_2(\hat{r}_3) &= \hat{r}_2, & \Delta_2(\hat{r}_4) &= \hat{r}_1, \\ \Delta_2(\hat{r}_5) &= \hat{r}_8, & \Delta_2(\hat{r}_6) &= \hat{r}_7, & \Delta_2(\hat{r}_7) &= \hat{r}_6, & \Delta_2(\hat{r}_8) &= \hat{r}_5; \end{aligned}$$

and

$$\begin{aligned} \Delta_1(\hat{r}_1) &= \hat{r}_3 + 1, & \Delta_1(\hat{r}_2) &= \hat{r}_4, & \Delta_1(\hat{r}_3) &= \hat{r}_1, & \Delta_1(\hat{r}_4) &= \hat{r}_2, \\ \Delta_1(\hat{r}_5) &= \hat{r}_7, & \Delta_1(\hat{r}_6) &= \hat{r}_8, & \Delta_1(\hat{r}_7) &= \hat{r}_5, & \Delta_1(\hat{r}_8) &= \hat{r}_6. \end{aligned}$$

In the vectorial case,  $\alpha$ -superhomogeneity is expressed as

$$\frac{1 + F_1F_2}{2} \cdot S^\alpha \Xi_{(Q_1, Q_2)}(A_1, A_2) = \frac{1 + F_1F_2}{2} \cdot \Xi_{(Q_1, Q_2)}((1 + t_1)A_1, (1 + t_2)A_2);$$

and, in the (pseudo)scalar case, it can be defined as

$$\frac{1 + F_1F_2}{2} \cdot (1 + \frac{\alpha}{2}t_1 + \frac{\alpha}{2}t_2) \cdot \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t_1)A_1, (1 + t_2)A_2).$$

Then,  $\alpha$ -superhomogeneity corresponds to the collection of relations

$$\hat{p}_{2,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,2,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,2}^{[s]} = \alpha\hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_2\hat{p})_{k_1,k_2,\dots,k_r}^{[s]};$$

where  $(\tilde{\Delta}_2\hat{p})_{k_1,k_2,\dots,k_r}^{[s]}$  is an appropriate linear combination of various  $\hat{p}_{l_1,l_2,\dots,l_r}^{[s]}$ . In particular, in the Clifford conservative case, this implies  $\hat{p}_2^{[s]} = \alpha$ .

In the vectorial case,  $\alpha$ -super-equiaffinity is expressed as

$$\frac{1 - F_1F_2}{2} \cdot S^\alpha \Xi_{(Q_1, Q_2)}(A_1, A_2) = \frac{1 - F_1F_2}{2} \cdot \Xi_{(Q_1, Q_2)}((1 + t_1)A_1, (1 + t_2)A_2);$$

and, in the (pseudo)scalar case, it can be defined as

$$\frac{1 - F_1F_2}{2} \cdot (1 + \frac{\alpha}{2}t_1 - \frac{\alpha}{2}t_2) \cdot \Xi_{(Q_1, Q_2)}(A_1, A_2) = \Xi_{(Q_1, Q_2)}((1 + t_1)A_1, (1 + t_2)A_2).$$

Then,  $\alpha$ -super-equiaffinity corresponds to the collection of relations

$$\hat{p}_{1,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,1,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,1}^{[s]} = (-1^{[s]})\alpha\hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_1\hat{p})_{k_1,k_2,\dots,k_r}^{[s]};$$

where  $(\tilde{\Delta}_1\hat{p})_{k_1,k_2,\dots,k_r}^{[s]}$  is an appropriate linear combination of various  $\hat{p}_{l_1,l_2,\dots,l_r}^{[s]}$ . In particular, in the Clifford conservative case, this implies  $\hat{p}_1^{[s]} = (-1^{[s]})\alpha$ .

At this point, one may wonder if there are similar scaling properties related to certain derivations  $\Delta_6, \Delta_7, \Delta_8$ . This is indeed the case, even if not so interesting.

**3.5.  $\alpha$ -Conjugation-invariance: skew-homogeneous ( $\underline{\text{CH}}^\alpha$ ) aspects.** Let  $t$  be an infinitesimal variable anticommuting with the  $Q_i, R_i$ , as in 3.3. Then there is a conjugation action given by

$$\text{CH}^\alpha(X) := (1 + \frac{\alpha}{2}t)X(1 - \frac{\alpha}{2}t).$$

Now, skew-homogeneous  $\alpha$ -conjugation invariance for  $\Xi$  means

$$\text{CH}^\alpha \Xi(A_1, A_2) = \Xi(\text{CH}^1 A_1, \text{CH}^1 A_2),$$

where  $\text{CH}^\alpha$  is meant componentwise in the vectorial case.

It turns out that  $\text{CH}^\alpha(X)$  induces a derivation  $\Delta_6$  given by

$$\begin{aligned} \Delta_6(\hat{r}_1) &= \hat{r}_8, & \Delta_6(\hat{r}_2) &= \hat{r}_7, & \Delta_6(\hat{r}_3) &= \hat{r}_6, & \Delta_6(\hat{r}_4) &= \hat{r}_5, \\ \Delta_6(\hat{r}_5) &= \hat{r}_4, & \Delta_6(\hat{r}_6) &= \hat{r}_3 + 1, & \Delta_6(\hat{r}_7) &= \hat{r}_2, & \Delta_6(\hat{r}_8) &= \hat{r}_1. \end{aligned}$$

Then, skew-homogeneous  $\alpha$ -conjugation-invariance corresponds to the collection of relations

$$\hat{p}_{6,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,6,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,6}^{[s]} = \alpha \hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_6 \hat{p})_{k_1,k_2,\dots,k_r}^{[s]}.$$

In particular, in the Clifford conservative case, it yields  $\hat{p}_6^{[s]} = \alpha$ . On the other hand, the naturality / conjugation invariance property always implies this scaling with  $\alpha = 0$  in the (pseudo)scalar case and with  $\alpha = 1$  in the vectorial case.

**3.6.  $\alpha$ -Conjugation-invariance: superhomogeneous ( $\underline{\text{CSH}}^\alpha$ ) and super-equiaffine ( $\underline{\text{CSE}}^\alpha$ ) aspects.** Let  $t_1$  and  $t_2$  be infinitesimal variables as in 3.4. Then there is a conjugation action given by

$$\text{CS}^\alpha(X) := (1 + \frac{\alpha}{2}(t_1 + t_2))X(1 - \frac{\alpha}{2}(t_1 + t_2)).$$

In the vectorial case, superhomogeneous  $\alpha$ -conjugation-invariance for  $\Xi$  means

$$\frac{1 + F_1 F_2}{2} \text{CS}^\alpha \Xi(A_1, A_2) = \frac{1 + F_1 F_2}{2} \Xi(\text{CS}^1 A_1, \text{CS}^1 A_2);$$

where  $\text{CS}^\alpha$  is meant componentwise ; and in the (pseudo)scalar case it is defined by

$$\frac{1 + F_1 F_2}{2} (1 + \frac{\alpha}{2}t_1 + \frac{\alpha}{2}t_2) \text{CS}^\alpha \Xi(A_1, A_2) = \frac{1 + F_1 F_2}{2} \Xi(\text{CS}^1 A_1, \text{CS}^1 A_2).$$

In the vectorial case, super-equiaffine  $\alpha$ -conjugation-invariance for  $\Xi$  means

$$\frac{1 - F_1 F_2}{2} \text{CS}^\alpha \Xi(A_1, A_2) = \frac{1 - F_1 F_2}{2} \Xi(\text{CS}^1 A_1, \text{CS}^1 A_2);$$

and in the (pseudo)scalar case it is defined by

$$\frac{1 - F_1 F_2}{2} (1 - \frac{\alpha}{2}t_1 + \frac{\alpha}{2}t_2) \text{CS}^\alpha \Xi(A_1, A_2) = \frac{1 - F_1 F_2}{2} \Xi(\text{CS}^1 A_1, \text{CS}^1 A_2).$$

There are induced derivations given by

$$\begin{aligned} \Delta_7(\hat{r}_1) &= \hat{r}_5, & \Delta_7(\hat{r}_2) &= \hat{r}_6, & \Delta_7(\hat{r}_3) &= \hat{r}_7, & \Delta_7(\hat{r}_4) &= \hat{r}_8, \\ \Delta_7(\hat{r}_5) &= \hat{r}_1, & \Delta_7(\hat{r}_6) &= \hat{r}_2, & \Delta_7(\hat{r}_7) &= \hat{r}_3 + 1, & \Delta_7(\hat{r}_8) &= \hat{r}_4; \end{aligned}$$

and

$$\begin{aligned} \Delta_8(\hat{r}_1) &= \hat{r}_6, & \Delta_8(\hat{r}_2) &= \hat{r}_5, & \Delta_8(\hat{r}_3) &= \hat{r}_8, & \Delta_8(\hat{r}_4) &= \hat{r}_7, \\ \Delta_8(\hat{r}_5) &= \hat{r}_2, & \Delta_8(\hat{r}_6) &= \hat{r}_1, & \Delta_8(\hat{r}_7) &= \hat{r}_4, & \Delta_8(\hat{r}_8) &= \hat{r}_3 + 1. \end{aligned}$$

Then superhomogeneous  $\alpha$ -conjugation-invariance corresponds to the collection of relations

$$\hat{p}_{7,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,7,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,7}^{[s]} = \alpha \hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_7 \hat{p})_{k_1,k_2,\dots,k_r}^{[s]};$$

and super-equiaffine  $\alpha$ -conjugation-invariance corresponds to the collection of relations

$$\hat{p}_{8,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,8,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,8}^{[s]} = (-1^{[s]})\alpha\hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_8\hat{p})_{k_1,k_2,\dots,k_r}^{[s]}.$$

In particular, in the Clifford conservative case, superhomogeneity implies  $\hat{p}_7^{[s]} = \alpha$ . However, the naturality condition always implies this scaling condition with  $\alpha = 0$  in the scalar case,  $\alpha = 1$  in the vectorial case, and  $\alpha = 2$  in the pseudoscalar case. In the super-equiaffine case,  $\hat{p}_8^{[s]} = (-1^{[s]})\alpha$  should hold. Naturality always implies this scaling condition with  $\alpha = 0$  in the (pseudo)scalar case and  $\alpha = 1$  in the vectorial case.

**3.7. Summary and general patterns.** In this section we have studied some scaling invariances adapted to the mixed basis. Then every element of the mixed basis corresponds to a scaling invariance. Regarding their behaviour, the variables of the mixed basis can be: homogeneous ( $\{2, 3, 6, 7\}$ ) or equiaffine ( $\{1, 4, 5, 8\}$ ); ordinary ( $\{3, 4, 5, 6\}$ ) or super ( $\{1, 2, 7, 8\}$ ); straight ( $\{1, 2, 3, 4\}$ ) or skew ( $\{5, 6, 7, 8\}$ ). In fact, every  $\hat{r}_i$  corresponds to a character, a  $\pm 1$  sequence, according to formula (5). We can define a group structure  $*$  on the ciphers  $\{1, \dots, 8\}$  corresponding to the positionwise multiplication of the characters. Here ‘3’ turns out to be the identity element (as it corresponds to the sequence containing only 1’s). This “character group”  $\mathfrak{C}$  is isomorphic to  $(Z_2)^3$  with multiplication table

$*$	1	2	3	4	5	6	7	8
1	3	4	1	2	7	8	5	6
2	4	3	2	1	8	7	6	5
3	1	2	3	4	5	6	7	8
4	2	1	4	3	6	5	8	7
5	7	8	5	6	3	4	1	2
6	8	7	6	5	4	3	2	1
7	5	6	7	8	1	2	3	4
8	6	5	8	7	2	1	4	3

Then every cipher  $i$  induces a derivation  $\Delta_i$  on  $(R/Q)$  defined by

$$\Delta_i(\hat{r}_j) = \delta_{i,j}1 + \hat{r}_{i*j}.$$

Using this  $\Delta_i$ , we can effectively write down the  $\hat{r}_i$ -scaling-invariance in terms of expansion coefficients as

$$\hat{p}_{i,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,i,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,i}^{[s]} = (-1^{[s]})^{\delta_{i \in \{1,4,5,8\}}} \alpha \hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_i \hat{p})_{k_1,k_2,\dots,k_r}^{[s]},$$

where

$$(\tilde{\Delta}_i \hat{p})_{k_1,k_2,\dots,k_r}^{[s]} = \hat{p}_{i*k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,i*k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,i*k_r}^{[s]}.$$

In the Clifford conservative case  $\hat{p}^{[0]} = 1 / \hat{p}^{[1]} = \hat{p}^{[2]} = 1 / \hat{p}^{[12]} = 1$ , the degrees of homogeneity can be recovered from  $\hat{p}_i^{[s]}$ . For  $i \in \{6, 7, 8\}$ , however, scaling invariance is always provided by naturality; hence not very interesting, although it retains some interest for pointed expansions.

**3.8.  $\alpha$ -circular ( $\underline{\mathbf{XE}}^\alpha$ ) and  $\alpha$ -skew-circular ( $\underline{\mathbf{CXE}}^\alpha$ ) invariance.** These are invariances associated to the group algebra elements  $\hat{4} = \frac{1}{2}(\hat{4} + \hat{5}) \in \mathbb{R}\mathfrak{C}$  and  $\hat{5} = \frac{1}{2}(\hat{4} - \hat{5}) \in \mathbb{R}\mathfrak{C}$ ; where the elements of  $\mathfrak{C}$  are denoted by  $\hat{1}, \dots, \hat{8}$ , in order to avoid confusion.

Then  $\alpha$ -circular ( $\tilde{r}_4$ -scaling) invariance ( $\underline{\mathbf{XE}}^\alpha$ ) is expressed as  $\frac{1}{2}((S4)|_{\alpha=\alpha} + (S5)|_{\alpha=\alpha})$ ; and  $\alpha$ -skew-circular ( $\tilde{r}_5$ -scaling) invariance ( $\underline{\mathbf{CXE}}^\alpha$ ) is expressed as  $\frac{1}{2}((S4)|_{\alpha=\alpha} - (S5)|_{\alpha=-\alpha})$ ; where (S4) and (S5) refer back to earlier equations. In the Clifford conservative case,  $\tilde{r}_4$ -scaling invariance implies  $\hat{p}_4^{[s]} = (-1^{[s]})\alpha$ , and  $\tilde{r}_5$ -scaling invariance implies  $\hat{p}_5^{[s]} = (-1^{[s]})\alpha$ .

Unfortunately, scaling invariances (in themselves) are rather weak properties.

## 4. HYPERSCALING

4.1. We say that the FQ operation  $\Xi$  satisfies the hyperscaling property of type  $(J, L, \alpha, \beta)$  in variable  $\hat{r}_h$ , component  $[s]$ , if in its expansion relative to the mixed base, the “decay” identities

$$\begin{aligned}\hat{p}_h^{[s]} &= (\alpha + \beta)\hat{p}^{[s]} \\ \hat{p}_{h,j,\dots}^{[s]} &= \alpha\hat{p}_{j,\dots}^{[s]} - \frac{1}{2}J\hat{p}_{6*h*j,\dots}^{[s]} + (-\frac{1}{2} - L)\hat{p}_{h*j,\dots}^{[s]} \\ \hat{p}_{\dots,i,h}^{[s]} &= \frac{1}{2}J\hat{p}_{\dots,i*h*6}^{[s]} + (-\frac{1}{2} + L)\hat{p}_{\dots,i*h}^{[s]} + \beta\hat{p}_{\dots,i}^{[s]} \\ \hat{p}_{\dots,i,h,j,\dots}^{[s]} &= \frac{1}{2}J\hat{p}_{\dots,i*h*6,j,\dots}^{[s]} + (-\frac{1}{2} + L)\hat{p}_{\dots,i*h,j,\dots}^{[s]} - \frac{1}{2}J\hat{p}_{\dots,i,6*h*j,\dots}^{[s]} + (-\frac{1}{2} - L)\hat{p}_{\dots,i,h*j,\dots}^{[s]}\end{aligned}$$

hold. Summing up the appropriate terms, we see that hyperscaling of type  $(J, L, \alpha, \beta)$  in variable  $\hat{r}_h$ , component  $[s]$  implies an  $(\alpha + \beta)$ -scaling rule in variable  $\hat{r}_h$ , component  $[s]$ ,

$$\hat{p}_{h,k_1,k_2,\dots,k_r}^{[s]} + \hat{p}_{k_1,h,k_2,\dots,k_r}^{[s]} + \dots + \hat{p}_{k_1,k_2,\dots,k_r,h}^{[s]} = (\alpha + \beta)\hat{p}_{k_1,k_2,\dots,k_r}^{[s]} - (\tilde{\Delta}_h\hat{p})_{k_1,k_2,\dots,k_r}^{[s]}.$$

Hyperscaling allows us to eliminate coefficients with indices  $h$  in the expansion relative to the mixed base. A related definition is as follows. We say that FQ operation  $\Xi$  satisfies character degeneracy with  $\pm 1$ , if in its expansion relative to the mixed base, the identities

$$\hat{p}_{\dots,i,\dots}^{[s]} = \pm \hat{p}_{\dots,i*6,\dots}^{[s]}$$

hold.

4.2. **Theorem.** (a) An FQ operation satisfies conjugation invariance, i. e. naturality, if and only if in its expansion relative to the mixed base, it satisfies

$$(C6) \quad \text{hyperscaling of type } (0, 0, \alpha_6^{[s]}, \beta_6^{[s]}) \text{ in } \hat{r}_6, \text{ component } [s],$$

$$(C7) \quad \text{hyperscaling of type } (1, 0, \alpha_7^{[s]}, \beta_7^{[s]}) \text{ in } \hat{r}_7, \text{ component } [s],$$

$$(C8) \quad \text{hyperscaling of type } (-1, 0, \alpha_8^{[s]}, \beta_8^{[s]}) \text{ in } \hat{r}_8, \text{ component } [s];$$

where, for scalar operations,

$$(\alpha_6^{[0]}, \beta_6^{[0]}) = (\frac{1}{2}, -\frac{1}{2}), \quad (\alpha_7^{[0]}, \beta_7^{[0]}) = (1, -1), \quad (\alpha_8^{[0]}, \beta_8^{[0]}) = (0, 0);$$

for vectorial operations,

$$\begin{aligned}(\alpha_6^{[1]}, \beta_6^{[1]}) &= (\frac{1}{2}, \frac{1}{2}), & (\alpha_7^{[1]}, \beta_7^{[1]}) &= (1, 0), & (\alpha_8^{[1]}, \beta_8^{[1]}) &= (0, 1), \\ (\alpha_6^{[2]}, \beta_6^{[2]}) &= (\frac{1}{2}, \frac{1}{2}), & (\alpha_7^{[2]}, \beta_7^{[2]}) &= (1, 0), & (\alpha_8^{[2]}, \beta_8^{[2]}) &= (0, -1);\end{aligned}$$

and for pseudoscalar operations

$$(\alpha_6^{[12]}, \beta_6^{[12]}) = (\frac{1}{2}, -\frac{1}{2}), \quad (\alpha_7^{[12]}, \beta_7^{[12]}) = (1, 1), \quad (\alpha_8^{[12]}, \beta_8^{[12]}) = (0, 0).$$

(b) The vectorial FQ operation is bivariant, if and only if it is symmetrically bivariant, if and only if, in addition to (C6)–(C9), it satisfies

$$(C45') \quad \text{character degeneracy with } +1.$$

(Symmetric) bivariance is inconsistent for (pseudo)scalar FQ operations except for the identically zero operation.

(b') The vectorial FQ operation is antivariant, if and only if it is symmetrically antivariant, if and only if, in addition to (C6)–(C9), it satisfies

$$(C45'') \quad \text{character degeneracy with } -1.$$



(Symmetric) antivariance is inconsistent for (pseudo)scalar FQ operations except for the identically zero operation.

(c) The (pseudo)scalar FQ operation is left-variant, if and only if it is symmetrically left-variant, if and only if, in addition to (C6)–(C9), it satisfies

$$(C45') \quad \text{character degeneracy with } +1.$$

(Symmetric) left-variance is inconsistent for vectorial FQ operations except for the identically zero operation.

(c') The (pseudo)scalar FQ operation is right-variant, if and only if it is symmetrically right-variant, if and only if, in addition to (C6)–(C9), it satisfies

$$(C45'') \quad \text{character degeneracy with } -1.$$

(Symmetric) right-variance is inconsistent for vectorial FQ operations except for the identically zero operation.

*Proof.* We consider  $\Xi$  on the perturbation  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$  of the Clifford system  $(Q_1, Q_2)$ .

(a) When we check conjugation invariance formally, it is sufficient to check infinitesimally, i. e. with respect to elements  $1 + \theta$ , where  $\theta \mathfrak{A} \theta = 0$ . Even so, we can decompose  $\theta$  into (anti)symmetric parts with respect to  $Q_1, Q_2$ . Let  $C$  denote the  $x \mapsto (1 + \theta)x(1 - \theta)$  conjugation action.

If  $\theta$  commutes with  $Q_1$  and  $Q_2$ , then conjugation invariance holds automatically.

If  $\theta$  anticommutes with  $Q_1$  and  $Q_2$ , then let  $\hat{\varrho}_1, \dots, \hat{\varrho}_8$  be the mixed base decomposition of  $C(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . By direct computation we find that

$$\hat{\rho}_i = \hat{r}_i + 2\delta_{i,6}\theta + \theta\hat{r}_{i*6} + \hat{r}_{i*6}\theta.$$

Then, in terms of the power series expansion, conjugation invariance means

$$\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8)Q^{[s]} = (1 + \theta)\hat{f}_s(\hat{r}_1, \dots, \hat{r}_8)Q^{[s]}(1 - \theta).$$

Considering the terms which are of order 1 in  $\theta$ , taking into account the (anti)commutation rules of  $\theta$  and  $Q^{[s]}$ , and the noncommutativity of the power series; the equality above translates to (C6).

If  $\theta$  anticommutes with  $Q_1Q_2$ , then it can be assumed that  $\theta = \theta_0F_1 + \theta_0F_2$ , where  $F_i^2 = 1$ ,  $F_i$  anticommutes with  $Q_i$ ,  $F_i$  commutes with  $Q_{1-i}$  and the  $\hat{r}_j$ , and  $\theta_0$  commutes with  $Q_j$ . Again, let  $\hat{\varrho}_1, \dots, \hat{\varrho}_8$  be the mixed base decomposition of  $C(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . By direct computation we find that

$$\frac{1 + F_1F_2}{2}\hat{\rho}_i = \frac{1 + F_1F_2}{2}(\hat{r}_i + 2\delta_{i,7}\theta_0F_1 + \theta_0F_1\hat{r}_{i*7} + \theta_0F_1\hat{r}_{i*2} + \hat{r}_{i*7}\theta_0F_1 - \hat{r}_{i*2}\theta_0F_1)$$

and

$$\frac{1 - F_1F_2}{2}\hat{\rho}_i = \frac{1 - F_1F_2}{2}(\hat{r}_i + 2\delta_{i,8}\theta_0F_1 + \theta_0F_1\hat{r}_{i*8} - \theta_0F_1\hat{r}_{i*1} + \hat{r}_{i*7}\theta_0F_1 + \hat{r}_{i*1}\theta_0F_1).$$

Then the equalities

$$\frac{1 + F_1F_2}{2}\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8)Q^{[s]} = \frac{1 + F_1F_2}{2}(1 + \theta)\hat{f}_s(\hat{r}_1, \dots, \hat{r}_8)Q^{[s]}(1 - \theta)$$

and

$$\frac{1 - F_1F_2}{2}\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8)Q^{[s]} = \frac{1 - F_1F_2}{2}(1 + \theta)\hat{f}_s(\hat{r}_1, \dots, \hat{r}_8)Q^{[s]}(1 - \theta)$$

compared with

$$\frac{1 + F_1F_2}{2}\theta = \frac{1 + F_1F_2}{2} \cdot 2\theta_0F_1 \quad \text{and} \quad \frac{1 - F_1F_2}{2}\theta = 0$$

yield (C7) and (C8).

(b) When we extend to bivarience, it is sufficient to check infinitesimal bivarience, i. e. with respect to elements  $1 + \theta$ , where  $\theta \mathfrak{A} \theta = 0$ ; and we start checking out bivarience with respect to the symmetric bivarience action  $B$  given by  $x \mapsto (1 + \theta)x(1 + \theta)$ . Again, we can decompose  $\theta$  into (anti)symmetric parts with respect to  $Q_1, Q_2$ .

If  $\theta$  anticommutes with  $Q_1$  and  $Q_2$ , then let  $\hat{\rho}_1, \dots, \hat{\rho}_8$  be the mixed base decomposition of  $B(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . By direct computation we find that

$$\hat{\rho}_i = \hat{r}_i + \theta \hat{r}_{6*i} - \hat{r}_{i*6} \theta.$$

Then, in terms of the power series expansion, symmetric bivarience means

$$\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8) Q^{[s]} = (1 + \theta) \hat{f}_s(\hat{r}_1, \dots, \hat{r}_8) Q^{[s]} (1 + \theta).$$

Considering the terms which are of order 1 in  $\theta$ , taking into account the (anti)commutation rules of  $\theta$  and  $Q^{[s]}$ , and the noncommutativity of the power series; the equality above translates to the collection of relations

$$\hat{p}_{\dots, i, 6*j, \dots}^{[s]} = \hat{p}_{\dots, i*6, j, \dots}^{[s]};$$

and for (pseudo)scalar operations

$$p^{[s]} = 0, \quad \hat{p}_{6*i, \dots}^{[s]} = \hat{p}_{i, \dots}^{[s]}, \quad \hat{p}_{\dots, j*6}^{[s]} = -\hat{p}_{\dots, j}^{[s]};$$

and for vectorial operations

$$\hat{p}_{6*i, \dots}^{[s]} = \hat{p}_{i, \dots}^{[s]}, \quad \hat{p}_{\dots, j*6}^{[s]} = \hat{p}_{\dots, j}^{[s]}.$$

One can see that for (pseudo)scalar operations this implies the vanishing of all coefficients, while for vectorial operations, it implies (C45'). For the rest of (infinitesimal) symmetric bivarience, we show that it is equivalent to

(C3') hyperscaling of type  $(0, 0, \alpha_3^{[s]}, \beta_3^{[s]})$  in  $\hat{r}_3$ , component  $[s]$ ,

(C2') hyperscaling of type  $(1, 0, \alpha_2^{[s]}, \beta_2^{[s]})$  in  $\hat{r}_2$ , component  $[s]$ ,

(C1') hyperscaling of type  $(-1, 0, \alpha_1^{[s]}, \beta_1^{[s]})$  in  $\hat{r}_1$ , component  $[s]$ ,

where

$$\begin{aligned} (\alpha_3^{[1]}, \beta_3^{[1]}) &= (\tfrac{1}{2}, \tfrac{1}{2}), & (\alpha_2^{[1]}, \beta_2^{[1]}) &= (1, 0), & (\alpha_1^{[1]}, \beta_1^{[1]}) &= (0, 1), \\ (\alpha_3^{[2]}, \beta_3^{[2]}) &= (\tfrac{1}{2}, \tfrac{1}{2}), & (\alpha_2^{[2]}, \beta_2^{[2]}) &= (1, 0), & (\alpha_1^{[2]}, \beta_1^{[2]}) &= (0, -1). \end{aligned}$$

If  $\theta$  commutes with  $Q_1$  and  $Q_2$ , then let  $\hat{\rho}_1, \dots, \hat{\rho}_8$  be the mixed base decomposition of  $B(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . By direct computation, we find that

$$\hat{\rho}_i = \hat{r}_i + 2\delta_{i,3}\theta + \theta \hat{r}_i + \hat{r}_i \theta.$$

Then, in terms of the power series expansion, symmetric bivarience means

$$\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8) Q^{[s]} = (1 + \theta) \hat{f}_s(\hat{r}_1, \dots, \hat{r}_8) Q^{[s]} (1 + \theta).$$

Considering the terms which are of order 1 in  $\theta$ , taking into account the (anti)commutation rules of  $\theta$  and  $Q^{[s]}$ , and the noncommutativity of the power series; the equality above translates to (C3').

If  $\theta$  anticommutes with  $Q_1 Q_2$ , then it can be assumed that  $\theta = \theta_0 F_1 + \theta_0 F_2$ , where  $F_i^2 = 1$ ,  $F_i$  anticommutes with  $Q_i$ ,  $F_i$  commutes with  $Q_{1-i}$  and the  $\hat{r}_j$ , and  $\theta_0$  commutes with  $Q_j$ . Again, let  $\hat{\rho}_1, \dots, \hat{\rho}_8$  be the mixed base decomposition of  $B(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . By direct computation we find that

$$\frac{1 + F_1 F_2}{2} \hat{\rho}_i = \frac{1 + F_1 F_2}{2} (\hat{r}_i + 2\delta_{i,2}\theta_0 F_1 + \theta_0 F_1 \hat{r}_{i*7} + \theta_0 F_1 \hat{r}_{i*2} - \hat{r}_{i*7} \theta_0 F_1 + \hat{r}_{i*2} \theta_0 F_1)$$

and

$$\frac{1 - F_1 F_2}{2} \hat{\rho}_i = \frac{1 - F_1 F_2}{2} (\hat{r}_i + 2\delta_{i,8}\theta_0 F_2 - \theta_0 F_2 \hat{r}_{i*8} + \theta_0 F_2 \hat{r}_{i*1} + \hat{r}_{i*7}\theta_0 F_2 + \hat{r}_{i*1}\theta_0 F_2).$$

Then the equalities

$$\frac{1 + F_1 F_2}{2} \hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8) Q^{[s]} = \frac{1 + F_1 F_2}{2} (1 + \theta) \hat{f}_s(\hat{r}_1, \dots, \hat{r}_8) Q^{[s]} (1 - \theta)$$

and

$$\frac{1 - F_1 F_2}{2} \hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8) Q^{[s]} = \frac{1 - F_1 F_2}{2} (1 + \theta) \hat{f}_s(\hat{r}_1, \dots, \hat{r}_8) Q^{[s]} (1 - \theta)$$

yield (C2') and (C1').

However, under character degeneracy with +1, (C3') is equivalent to (C6); (C2') is equivalent to (C7), (C1') is equivalent to (C8); so symmetric bivariance also implies conjugation invariance. (Infinitesimal) conjugation invariance and (infinitesimal) symmetric bivariance, however, implies full (infinitesimal) bivariance. This argument also shows that (C45') is sufficient to provide bivariance in addition to (C6)–(C8).

(b') We can proceed in similar manner. If  $\theta$  anticommutes with  $Q_1$  and  $Q_2$ , then let  $\hat{\varrho}_1, \dots, \hat{\varrho}_8$  be the mixed base decomposition of  $B(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . We can apply the same analysis as before. Then, in terms of the power series expansion, symmetric antivariance means

$$\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8) Q^{[s]} = (1 - \theta) \hat{f}_s(\hat{r}_1, \dots, \hat{r}_8) Q^{[s]} (1 - \theta).$$

Considering the terms which are of order 1 in  $\theta$ , taking into account the (anti)commutation rules of  $\theta$  and  $Q^{[s]}$ , and the noncommutativity of the power series; the equality above translates to the collection of relations

$$\hat{p}_{\dots, i, 6*j, \dots}^{[s]} = \hat{p}_{\dots, i*6, j, \dots}^{[s]};$$

and for (pseudo)scalar operations

$$p^{[s]} = 0, \quad \hat{p}_{6*i, \dots}^{[s]} = -\hat{p}_{i, \dots}^{[s]}, \quad \hat{p}_{\dots, j*6}^{[s]} = \hat{p}_{\dots, j}^{[s]};$$

and for vectorial operations

$$\hat{p}_{6*i, \dots}^{[s]} = -\hat{p}_{i, \dots}^{[s]}, \quad \hat{p}_{\dots, j*6}^{[s]} = -\hat{p}_{\dots, j}^{[s]}.$$

One can see that for (pseudo)scalar operations, this implies the vanishing of all coefficients, while for vectorial operations, it implies (C45''). Then the further aspects of symmetric antivariance can be encoded by

(C3'') hyperscaling of type  $(0, 0, -\alpha_3^{[s]}, -\beta_3^{[s]})$  in  $\hat{r}_3$ , component  $[s]$ ,

(C2'') hyperscaling of type  $(1, 0, -\alpha_2^{[s]}, -\beta_2^{[s]})$  in  $\hat{r}_2$ , component  $[s]$ ,

(C1'') hyperscaling of type  $(-1, 0, -\alpha_1^{[s]}, -\beta_1^{[s]})$  in  $\hat{r}_1$ , component  $[s]$ .

The rest of the argument is analogous.

(c) Again, let  $\hat{\varrho}_1, \dots, \hat{\varrho}_8$  be the mixed base decomposition of  $B(A_1, A_2)$  with respect to  $(Q_1, Q_2)$ . In terms of the power series expansion, symmetric antivariance means

$$\hat{f}_s(\hat{\rho}_1, \dots, \hat{\rho}_8) Q^{[s]} = (1 + \theta) \hat{f}_s(\hat{r}_1, \dots, \hat{r}_8) Q^{[s]} (1 - \theta).$$

Then we can proceed as in (b).

(c') This is analogous to (b'). □

**4.3. Remark.** At first sight, the previous theorem is just plainly more informative the argument in 2.1, but this is not completely so. An advantage of 2.1 is that it proves that using (C6)–(C8) we are led to an unambiguous reduction in term of the indices 1, 2, 3, 4, 5 (in the mixed and circular bases).

**4.4. Corollary.** *One can compute the coefficients of the expansion of  $\underline{\mathcal{Q}}^{\text{Sy}}$  in the mixed basis, recursively, using*

- (i)  $\hat{p}^{[1]} = \hat{p}^{[2]} = 1$ ;
- (ii)  $\hat{p}_{\iota_1, \dots, \iota_r}^{[s]} = 0$  if  $\{\iota_1, \dots, \iota_r\}$  is a nonempty subset of  $\{1, \dots, 5\}$ ;
- (iii) the vectorial hyperscaling rules (C6), (C7), (C8) of the previous theorem.  $\square$

**4.5.** One can prove similar statements regarding the splitting and circular bases, too. Then, instead of  $*$ , one should use more complicated incidence matrices of indices. In particular, in case of the circular basis, character degeneracy yields  $\tilde{p}_{\dots, i, \dots}^{[s]} = \pm \tilde{p}_{\dots, i*6, \dots}^{[s]}$  for  $i \in \{1, 2, 3, 6, 7, 8\}$ ; and,  $\tilde{p}_{\dots, 5, \dots}^{[s]} = 0$  in the  $+1$  case, and  $\tilde{p}_{\dots, 4, \dots}^{[s]} = 0$  in the  $-1$  case.

**4.6. Theorem.** (b) *If  $\Xi$  is a bivariant vectorial FQ operation, then it is determined by the coefficients  $\tilde{p}_{4, \dots, 4}^{[s]}$ ,  $s \in \{1, 2\}$ , which can be prescribed arbitrarily.*

(b') *If  $\Xi$  is an antivariant vectorial FQ operation, then it is determined by the coefficients  $\tilde{p}_{5, \dots, 5}^{[s]}$ ,  $s \in \{1, 2\}$ , which can be prescribed arbitrarily.*

(c) *If  $\Xi$  is a left-variant (pseudo)scalar FQ operation, then it is determined by the coefficients  $\tilde{p}_{4, \dots, 4}^{[s]}$ , ( $s = 0$  or  $s = 12$ ) which can be prescribed arbitrarily.*

(c') *If  $\Xi$  is a right-variant (pseudo)scalar FQ operation, then it is determined by the coefficients  $\tilde{p}_{5, \dots, 5}^{[s]}$ , ( $s = 0$  or  $s = 12$ ) which can be prescribed arbitrarily.*

(The statement is also true using the mixed base, and then we do not even have worry whether 4 or 5 should be used as indices.)

*Proof.* (b) First considering the mixed basis, due to character degeneracy, we can pass to the denegerate base, where  $\hat{1} = \hat{8}, \hat{2} = \hat{7}, \hat{3} = \hat{7}, \hat{4} = \hat{5}$ . Then all we have to do is to impose (C6)–(C8). When we do this we reduce everything to coefficients  $\hat{p}_{4, \dots, 4}^{[s]}$ . The reduction to these terms using (C6)–(C8) leads to unambiguous results; or, said differently, it imposes no relations between the  $\hat{p}_{4, \dots, 4}^{[s]}$ , because we have that much freedom ( $r$  linear degrees of freedom up to order  $r$ ) even in a floating analytic expansion (discussed in [4] in the bivariant and antivariant cases, and left to the reader in the left- and right-variant cases). When we pass to the circular basis, we see that all the coefficients  $\tilde{p}_{\dots, 5, \dots}^{[s]}$  must vanish, so the setting is in fact reduced to the coefficients  $\tilde{p}_{4, \dots, 4}^{[s]}$ .

(b') First considering the mixed basis, due to character degeneracy, we can pass to the denegerate base, similarly. The accounting is just a little bit trickier due to the sign changes  $\hat{1} = -\hat{8}, \hat{2} = -\hat{7}, \hat{3} = -\hat{7}, \hat{4} = -\hat{5}$ . Again we arrive to reduction to the coefficients  $\hat{p}_{4, \dots, 4}^{[s]}$ . When we pass to the circular basis, we see that all the coefficients  $\tilde{p}_{\dots, 4, \dots}^{[s]}$  must vanish, so the setting is, in fact, reduced to the coefficients  $\tilde{p}_{5, \dots, 5}^{[s]}$ .

(c) and (c') can be proven analogously.  $\square$

**4.7. Example.** In [4] we have introduced the conform orthogonization procedure  $\mathcal{O}^{\text{fSy}}$ . We define the anticonform orthogonization procedure  $\mathcal{O}^{\text{afSy}}$  such that

$$\mathcal{O}^{\text{afSy}}(A_1, A_2)_i := -(\mathcal{O}^{\text{fSy}}(A_1, A_2)_i)^{-1};$$

i. e., relative to  $\mathcal{O}^{\text{fSy}}$ , we take the inverse times  $-1$  in every component.  $\mathcal{O}^{\text{fSy}}$  and  $\mathcal{O}^{\text{afSy}}$  are vectorial FQ operations.

We define the pseudoscalar operation left axis  $\mathcal{A}_L$  by

$$\mathcal{A}_L(A_1, A_2) := -\text{pol } A_1 A_2^{-1} = \text{pol } A_2 A_1^{-1};$$

and the right axis  $\mathcal{A}_R$  by

$$\mathcal{A}_R(A_1, A_2) := -\text{pol } A_1^{-1} A_2 = \text{pol } A_2^{-1} A_1.$$

These operations above are, in fact, analytic FQ operations. One can show that  $\mathcal{O}^{\text{fSy}}$  is bivariant,  $\mathcal{O}^{\text{afSy}}$  is antivariant,  $\mathcal{A}_L$  is left-variant,  $\mathcal{A}_R$  is right-variant; and they are all orthogonal invariant and Clifford conservative. In what follows, we will consider their formal restrictions (in notation: underlined). We remark that in their (formal) expansion

$$\begin{array}{lll} \mathcal{O}^{\text{fSy}} : & \tilde{\mathbf{P}}_0^{[1]} = [1] & \tilde{\mathbf{P}}_1^{[1]} = [1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1], \\ & \tilde{\mathbf{P}}_0^{[2]} = [1] & \tilde{\mathbf{P}}_1^{[2]} = [-1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ -1]; \\ \mathcal{O}^{\text{afSy}} : & \tilde{\mathbf{P}}_0^{[1]} = [1] & \tilde{\mathbf{P}}_1^{[1]} = [-1 \ -1 \ -1 \ 0 \ 0 \ 1 \ 1 \ 1], \\ & \tilde{\mathbf{P}}_0^{[2]} = [1] & \tilde{\mathbf{P}}_1^{[2]} = [1 \ -1 \ -1 \ 0 \ 0 \ 1 \ 1 \ -1]; \\ \mathcal{A}_L : & \tilde{\mathbf{P}}_0^{[12]} = [1] & \tilde{\mathbf{P}}_1^{[12]} = [0 \ 2 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0]; \\ \mathcal{A}_R : & \tilde{\mathbf{P}}_0^{[12]} = [1] & \tilde{\mathbf{P}}_1^{[12]} = [0 \ -2 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0]. \end{array}$$

**4.8. Theorem.** (a) If  $\Xi$  is a left-variant or right-variant scalar FQ operation which is orthogonal invariant such that  $\tilde{p}^{[0]} = 1$ , then

$$\Xi = \underline{1}.$$

(b) If  $\Xi$  is a bivariant vectorial FQ operation which is orthogonal invariant such that  $\tilde{p}^{[1]} = \tilde{p}^{[2]} = 1$ , then

$$\Xi = \alpha \cdot \underline{\text{Id}} + (1 - \alpha) \cdot \underline{\mathcal{O}^{\text{fSy}}},$$

where  $\tilde{p}_4^{[1]} = -\tilde{p}_4^{[2]} = \alpha$ .

(b') If  $\Xi$  is an antivariant vectorial FQ operation which is orthogonal invariant such that  $\tilde{p}^{[1]} = \tilde{p}^{[2]} = 1$ , then

$$\Xi = \underline{\mathcal{O}^{\text{afSy}}}.$$

(c) If  $\Xi$  is a left-variant pseudoscalar FQ operation which is orthogonal invariant such that  $\tilde{p}^{[12]} = 1$ , then

$$\Xi = \underline{\mathcal{A}_L}.$$

(c') If  $\Xi$  is a right-variant pseudoscalar FQ operation which is orthogonal invariant such that  $\tilde{p}^{[12]} = 1$ , then

$$\Xi = \underline{\mathcal{A}_R}.$$

*Proof.* (a) If  $\Xi$  is a left-variant scalar operation, then according to the previous theorem, the operation depends only on the collection of coefficients  $\tilde{p}_{4,\dots,4}^{[s]}$ . According to our prescriptions,  $\tilde{p}^{[0]} = 1$ ; while, according to Lemma 2.7, orthogonal invariance implies that  $\tilde{p}_{4,\dots,4}^{[s]} = 0$  if the number of lower indices is bigger than zero. This implies that there is at most one such operation. However, the choice  $\Xi = 1$  satisfies the requirements for such an operation.

A very similar argument applies if  $\Xi$  is a right-variant scalar operation, and in cases (b'), (c), (c'). Case (b) is a little bit different, because Lemma 2.7 does not tell about the vanishing of  $\tilde{p}_4^{[1]} = -\tilde{p}_4^{[2]}$  (the equality follows from symmetry). And indeed, the linear combination given in the statement of (b) allows arbitrary choice for  $\tilde{p}_4^{[1]} = -\tilde{p}_4^{[2]}$ .  $\square$

4.9. Sometimes it is useful to consider hyperscaling with respect to the variables  $\tilde{r}_4$  and  $\tilde{r}_5$ . This means with respect to  $\tilde{4} = \frac{1}{2}(\hat{4} + \hat{5}) \in \mathbb{R}\mathfrak{C}$  and  $\tilde{5} = \frac{1}{2}(\hat{4} - \hat{5}) \in \mathbb{R}\mathfrak{C}$ ; the corresponding equations are basically the sums and differences of  $\hat{r}_4$  and  $\hat{r}_5$ -scaling equations similarly to as in 3.8.

Hyperscaling constraints exhibit a structured and nontrivial behaviour, which we cannot discuss here. However, if it is said that scaling constraints are too weak, then it must be said that hyperscaling constraints are too restrictive.

## 5. FLOATING CLIFFORD CONSERVATIVITY

5.1. (vC') Floating Clifford conservativity: A vectorial FQ operation  $\Xi$  satisfies this property if it acts trivially on floating Clifford systems.

(vC'') Floating Clifford anticonservativity: A vectorial FQ operation  $\Xi$  satisfies this property if it inverts floating Clifford systems, and multiplies them by  $-1$ .

In order to deal with floating Clifford (anti)conservativity, we use the following

5.2. **Theorem.** *Suppose that  $(Q_1, Q_2)$  is the symmetric orthogonalization of  $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$ , i. e.  $\tilde{r}_6 = \tilde{r}_7 = \tilde{r}_8 = 0$  in the circular base.*

(a) *We claim: If  $(A_1, A_2)$  is floating Clifford system, then  $\tilde{r}_4$  and  $\tilde{r}_5$  can be expressed from  $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$ , by some fixed explicit power series  $\tilde{F}_4, \tilde{F}_5$ :*

$$\tilde{r}_4 = \tilde{F}_4(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) \equiv \tilde{r}_2\tilde{r}_1 + O((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)^{\geq 3});$$

$$\tilde{r}_5 = \tilde{F}_5(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) \equiv \tilde{r}_1\tilde{r}_2 + O((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)^{\geq 3}).$$

(b) *Conversely, in the general case, the terms  $\tilde{r}_4$  and  $\tilde{r}_5$  can be replaced by  $\tilde{F}_4(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$  and  $\tilde{F}_5(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$  in order to yield a floating Clifford system.*

*Similar statement holds in the mixed base.*

*Proof.* (a) From the equations  $(A_1 A_2^{-1})^2 = -1$  and  $(A_2^{-1} A_1)^2 = -1$  we obtain

$$\tilde{r}_4 = \tilde{r}_2\tilde{r}_1 - \tilde{r}_2\tilde{r}_4 + \tilde{r}_4\tilde{r}_2 + \tilde{r}_4\tilde{r}_3 - 2\tilde{r}_4\tilde{r}_4 + O((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5)^{\geq 3})$$

and

$$\tilde{r}_5 = \tilde{r}_1\tilde{r}_2 + \tilde{r}_2\tilde{r}_5 + \tilde{r}_3\tilde{r}_5 - \tilde{r}_5\tilde{r}_2 - 2\tilde{r}_5\tilde{r}_5 + O((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5)^{\geq 3}).$$

Iterating these expressions, we find expressions  $\tilde{F}_4, \tilde{F}_5$  as indicated.

(b) Consider the algebra generated by  $\underline{Q}_1, \underline{Q}_2, \underline{\tilde{r}}_1, \underline{\tilde{r}}_2, \underline{\tilde{r}}_3$  subject to the appropriate commutation relations. Let  $(A_1, A_2) = (1 + \underline{\tilde{r}}_2 + \frac{1}{2}\underline{\tilde{r}}_3) \cdot (\underline{Q}_1, \underline{Q}_2) \cdot (1 + \underline{\tilde{r}}_2 + \frac{1}{2}\underline{\tilde{r}}_3)$ , which is a floating Clifford system. Let  $(Q_1, Q_2)$  be its symmetric orthogonalization. Then one finds that

$$\tilde{r}_i = \underline{\tilde{r}}_i + O((\underline{\tilde{r}}_1, \underline{\tilde{r}}_2, \underline{\tilde{r}}_3)^{\geq 2}) \quad \text{for } i \in \{1, 2, 3\}.$$

This implies that  $\underline{\tilde{r}}_1, \underline{\tilde{r}}_2, \underline{\tilde{r}}_3$  and  $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$  can be expressed from each other; which proves that having a floating Clifford system implies no nontrivial relations for  $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$ , and in fact, their free prescribability.  $\square$

The process of (b) yields, in fact, an FQ operation  $\underline{\mathcal{F}}^{\text{Sy}}$  producing floating Clifford systems, compatible with  $\underline{\mathcal{Q}}^{\text{Sy}}$  (hence different from  $\underline{\mathcal{Q}}^{\text{fSy}}$ , and therefore not bivariant).

5.3. **Theorem.** *For a conjugation-invariant vectorial FQ operation, floating Clifford (anti-) conservativity is equivalent to a collections of relations*

$$\tilde{p}_{\iota_1, \dots, \iota_r}^{[s]} = \text{an inhomogeneous linear expression of } \tilde{p}_{\chi_1, \dots, \chi_h}^{[s]} \text{'s with } h < r,$$

where  $\{\iota_1, \dots, \iota_r\} \subset \{1, 2, 3\}$ ,  $s \in \{1, 2\}$ . Similar statement holds in the mixed base.

*Proof.* We know that a conjugation-invariant vectorial FQ operation can be encoded by formal power series  $\tilde{f}_s(\tilde{r}_1, \dots, \tilde{r}_5)$  with respect to the symmetric orthogonalization  $(Q_1, Q_2)$ . Considering  $(A_1, A_2) = ((1 + \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + \tilde{r}_4)Q_1, (1 - \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 - \tilde{r}_4)Q_2)$ , we see that floating Clifford conservativity (+) and anticonservativity (−) can be expressed by

$$\tilde{f}_1(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{F}_4(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3), \tilde{F}_5(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)) = \pm((1 + \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + \tilde{F}_4(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3))Q_1)^{\pm 1}Q_1^{-1},$$

$$\tilde{f}_2(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{F}_4(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3), \tilde{F}_5(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)) = \pm((1 - \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 - \tilde{F}_4(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3))Q_2)^{\pm 1}Q_2^{-1}.$$

As  $\tilde{F}_4, \tilde{F}_5$  are of higher degree, the equations expand as in the statement.  $\square$

We see that floating Clifford (anti)conservativity is a quite weak property ( $2 \cdot 3^r$  constraints compared to  $2 \cdot 5^r$  free parameters) although not trivial.

## 6. CLIFFORD PRODUCTIVITY

**6.1. (CP) Clifford productivity:** A vectorial FQ operation  $\Xi$  satisfies this property if it produces Clifford systems. A pseudoscalar FQ operation  $\Xi$  satisfies this property if it produces skew-involutions.

**(CP') Floating Clifford productivity:** A vectorial FQ operation  $\Xi$  satisfies this property if it produces floating Clifford systems.

One can deal with Clifford productivity as follows. First, it is reasonable to restrict to the Clifford conservative case; then one can use the fact that Clifford systems close to each other are conjugates of each other (in case of floating Clifford systems: translates of each other). Done carefully, one can organize the conjugation scheme such that it provides existence and unicity at the same time. Such an analysis was already considered in [4], here we give a more detailed account.

**6.2. Convention.** Suppose that  $\Xi$  is an FQ operation. If  $\Xi$  is (pseudo)scalar operation, then  $\Xi^{-1}$  denotes the inverse with respect to 1, regarding multiplication in  $\mathfrak{A}$ ; if  $\Xi$  is vectorial operation, then  $\Xi^{-1}$  denotes the inverse with respect to Id, regarding composition of FQ operations (with respect to the same base point as of  $\Xi$ ).

We will use the notation sFQ, vFQ, psFQ for the spaces of scalar, vectorial, or pseudoscalar FQ operations respectively. We will use the notation  $\text{sFQ}^{\leq r}, \text{sFQ}^{(r)}, \text{sFQ}^{\geq r}$  for those scalar FQ operations whose expansion terms in  $(R/Q)$  are in degrees  $\leq r$ , exactly in degree  $r$ , or in degrees  $\geq r$ , respectively; we use similar notion in the other cases, too.

In the case of a concrete FQ operation  $\Xi$ , let  $\Xi^{\leq r}, \Xi^{(r)}, \Xi^{\geq r}$  denote those FQ operations which we obtain from  $\Xi$  by restricting its expansion to the indicated orders.

**6.3. Lemma.** Consider the maps

$$\eta \circlearrowleft \text{sFQ}^{(r)} \xrightleftharpoons[\lambda]{\varkappa} \text{vFQ}^{(r)} \xrightleftharpoons[\bar{\lambda}]{\bar{\varkappa}} \text{sFQ}^{(r)} \oplus \text{psFQ}^{(r)} \oplus \text{sFQ}^{(r)} \circlearrowright \bar{\eta}$$

given by

$$\eta^{(r)} : U^{(r)} \mapsto (U^{(r)})_{Q_1 Q_2}^0;$$

$$\lambda^{(r)} : U^{(r)} \mapsto U^{(r)} \cdot (Q_1, Q_2) - (Q_1, Q_2) \cdot U^{(r)};$$

$$\varkappa^{(r)} : (V_1^{(r)}, V_2^{(r)}) \mapsto \frac{1}{2}(V_1^{(r)} Q_1^{-1})_Q^{10} + \frac{1}{4}(V_1^{(r)} Q_1^{-1})_Q^{11} + \frac{1}{2}(V_2^{(r)} Q_2^{-1})_Q^{01} + \frac{1}{4}(V_2^{(r)} Q_2^{-1})_Q^{11};$$

$$\bar{\lambda}^{(r)} : (V_1^{(r)}, V_2^{(r)}) \mapsto ([V_1^{(r)}, Q_1]_+, [V_2^{(r)}, Q_1]_+ + [V_1^{(r)}, Q_2]_+, [V_2^{(r)}, Q_2]_+);$$

$$\bar{\varkappa}^{(r)} : (W_1^{(r)}, W_{12}^{(r)}, W_2^{(r)}) \mapsto$$

$$\left( -\frac{1}{4}[W_1^{(r)}, Q_1]_+ - \frac{1}{8}[W_{12}^{(r)}, Q_1 Q_2]_+ \cdot Q_1, -\frac{1}{4}[W_2^{(r)}, Q_2]_+ + \frac{1}{8}[W_{12}^{(r)}, Q_1 Q_2]_+ \cdot Q_2 \right);$$

$$\bar{\eta}^{(r)} : (W_1^{(r)}, W_{12}^{(r)}, W_2^{(r)}) \mapsto$$

$$\left( (W_1^{(r)})_{Q_1}^1, \frac{1}{2}[(W_1^{(r)})_{Q_1}^0, Q_1 Q_2] + (W_{12}^{(r)})_{Q_1 Q_2}^1 - \frac{1}{2}[(W_1^{(r)})_{Q_1}^0, Q_1 Q_2], (W_2^{(r)})_{Q_2}^1 \right).$$

(Remark: This involves a symmetric “connection choice” for  $\varkappa^{(r)}, \bar{\varkappa}^{(r)}, \bar{\eta}^{(r)}$ .)

Then, we claim, there are equalities

$$\begin{aligned} \text{PreAmb}_Q^{(r)} &:= \text{im } \varkappa^{(r)} = \ker \eta^{(r)}, \\ \text{Amb}_Q^{(r)} &:= \text{im } \lambda^{(r)} = \ker \bar{\lambda}^{(r)}, \\ \text{CoAmb}_Q^{(r)} &:= \text{im } \bar{\varkappa}^{(r)} = \ker \varkappa^{(r)}, \\ \text{PreCoAmb}_Q^{(r)} &:= \text{im } \bar{\lambda}^{(r)} = \ker \bar{\eta}^{(r)}. \end{aligned}$$

Furthermore, the maps  $\varkappa^{(r)}$  and  $\lambda^{(r)}$  induce (inverse) bijections between  $\text{PreAmb}_Q^{(r)}$  and  $\text{Amb}_Q^{(r)}$ ; the maps  $\bar{\lambda}^{(r)}$  and  $\bar{\varkappa}^{(r)}$  induce (inverse) bijections between  $\text{CoAmb}_Q^{(r)}$  and  $\text{PreCoAmb}_Q^{(r)}$ ; leading to a splitting

$$\text{vFQ}^{(r)} = \text{Amb}_Q^{(r)} \oplus \text{CoAmb}_Q^{(r)}$$

with factors

$$\pi_Q^{(r)} = \lambda^{(r)} \circ \varkappa^{(r)} \quad \text{and} \quad \bar{\pi}_Q^{(r)} = \bar{\varkappa}^{(r)} \circ \bar{\lambda}^{(r)}.$$

*Proof.* The point is that elements  $V^{(r)} \in \text{Amb}_Q^{(r)}$  are of shape

$$(V^{(r)}/Q)_{\text{split}} = \begin{bmatrix} 0 & 0 & x_3^{(r)} & x_4^{(r)} & 0 & x_6^{(r)} & 0 & x_4^{(r)} \end{bmatrix},$$

in bijection to elements  $U^{(r)} \in \text{PreAmb}_Q^{(r)}$

$$U^{(r)} = \frac{1}{2}(x_3^{(r)} + x_4^{(r)} + x_6^{(r)});$$

and elements  $V^{(r)} \in \text{CoAmb}_Q^{(r)}$  are of shape

$$(V^{(r)}/Q)_{\text{split}} = \begin{bmatrix} x_1^{(r)} & x_2^{(r)} & 0 & x_4^{(r)} & x_5^{(r)} & 0 & x_7^{(r)} & -x_4^{(r)} \end{bmatrix},$$

in bijection to elements  $W^{(r)} \in \text{PreCoAmb}_Q^{(r)}$

$$W^{(r)} = (-2(x_1^{(r)} + x_2^{(r)}), (2x_2^{(r)} + 4x_4^{(r)} - 2x_7^{(r)})Q_1 Q_2, -2(x_5^{(r)} + x_7^{(r)})).$$

(But formally, we can just check some relations between  $\eta^{(r)}, \lambda^{(r)}, \varkappa^{(r)}, \bar{\lambda}^{(r)}, \bar{\varkappa}^{(r)}, \bar{\eta}^{(r)}$ .)  $\square$

**6.4. Theorem.** Suppose that  $\Psi$  is an Clifford productive, Clifford conservative, vectorial FQ operation. Then

$$(9) \quad \begin{aligned} \Psi(A_1, A_2) &= \dots (1 + U^{(3)})(1 + U^{(2)})(1 + U^{(1)}) \cdot (Q_1, Q_2) \cdot \\ &\quad \cdot (1 + U^{(1)})^{-1}(1 + U^{(2)})^{-1}(1 + U^{(3)})^{-1} \dots, \end{aligned}$$

where  $U^{(i)} \in \text{sFQ}^{(i)}$ . Furthermore, we can obtain a most economical choice for the  $U^{(r)}$  (yielding bijective correspondence to  $\Psi$ ) by choosing  $U^{(r)}$  from

$$\text{PreAmb}_Q^{(r)} = \{U^{(r)} \in \text{sFQ}^{(r)} : (U^{(r)})_{Q_1 Q_2}^0 = 0\}.$$



With this choice,  $U^{(r)}$  can be recovered from  $\Psi/\mathbf{vFQ}^{\geq r+1}$  by simple arithmetics.

*Proof.* First, on the constructive side: One can see that with arbitrary choice, (9) yields a Clifford productive operation  $\Psi$ . (Because it makes sense and it behaves so in every order.) Let  $\Psi^{(r)}$  be the version of the FQ operation when we use only  $U^{(2)}, \dots, U^{(r)}$ , but not further. Then

$$\Psi^{(r)}(A_1, A_2) - \Psi^{(r-1)}(A_1, A_2) = \Delta\Psi^{(r)} + O((R/Q)^{\geq r+1}),$$

where  $\Delta\Psi^{(r)}$  is  $r$ -homogeneous in  $(R/Q)$ . In fact,

$$\Delta\Psi^{(r)} = U^{(r)} \cdot (Q_1, Q_2) - (Q_1, Q_2) \cdot U^{(r)}.$$

This term describes exactly how much ambiguity arises we step up from order  $r-1$  to  $r$ . Such ambiguities form  $\text{Amb}_Q^{(r)}$ , and these ambiguities can be achieved using  $U^{(r)} \in \text{PreAmb}_Q^{(r)}$  (and there is a bijective correspondence).

However, in terms of ambiguities, one cannot do better even in the general case. Suppose that we have a Clifford conservative operation FQ operation from  $\mathbf{vFQ}^{\leq r-1}$ , such that it is a Clifford system modulo  $\mathbf{vFQ}^{\geq r}$ . Suppose that we manage to extend it to an FQ operation  $\Psi \in \mathbf{vFQ}^{\leq r}$ , which is a Clifford system modulo  $\mathbf{vFQ}^{\geq r+1}$ . But there is the possibility of getting another version  $\Psi'$  so that

$$\Psi' = \Psi + V^{(r)} + O((R/Q)^{\geq r+1}),$$

where  $V^{(r)} \in \mathbf{vFQ}^{(r)}$ . Then the Clifford system property implies

$$\begin{aligned} Q_1 V_1^{(r)} + V_1^{(r)} Q_1 &= 0, \\ Q_1 V_2^{(r)} + V_1^{(r)} Q_2 + Q_2 V_1^{(r)} + V_2^{(r)} Q_1 &= 0, \\ Q_2 V_2^{(r)} + V_2^{(r)} Q_2 &= 0, \end{aligned}$$

which is equivalent to  $V^{(r)} \in \text{Amb}_{(Q_1, Q_2)}^{(r)}$ . Proceeding inductively, we see that every Clifford productive, Clifford conservative, vectorial FQ operation occurs in form (9), and in fact, using the economical choice for the  $U^{(r)}$ . Then, in each step, we have a full involutive operation  $\Psi^{(r)}$ , and the modifier terms arithmetically determined as follows: If  $U^{(2)}, \dots, U^{(r-1)}$ , are already recovered, then so is  $\Psi^{(r-1)}$ , and  $U^{(r)} = \kappa_Q^{(r)}((\Psi - \Psi^{(r-1)})^{(r)})$ .  $\square$

**6.5. Remark.** We could have used the form

$$\begin{aligned} \Psi(A_1, A_2) &= (1 + U^{(1)})(1 + U^{(2)})(1 + U^{(3)}) \dots (Q_1, Q_2) \dots \\ &\dots (1 + U^{(3)})^{-1}(1 + U^{(2)})^{-1}(1 + U^{(1)})^{-1}; \end{aligned}$$

it leads to the same ambiguities at each level. In this latter form, unicity is even more transparent. Another possibility is to take the exponential version where  $1 + U^{(r)}$  is replaced by  $\exp U^{(r)}$ .

For the sake of the next theorem, we use the general notation  $(\Psi_1, \Psi_2) \odot (\Psi_1, \Psi_2) = (\Psi_1 \Psi_1, \Psi_1 \Psi_2 + \Psi_2 \Psi_1, \Psi_2 \Psi_2)$ .

**6.6. Theorem.** Consider a Clifford conservative, vectorial FQ operation  $\Psi$ . Then, we claim, the Clifford productivity of  $\Psi$  can be expressed by the equations

$$(10) \quad \bar{\pi}_Q^{(r)} \Psi^{(r)} + \bar{\kappa}_Q^{(r)}((\Psi^{\leq r-1} \odot \Psi^{\leq r-1})^{(r)}) = 0$$

( $r \geq 2$ ). That is, a collection of  $\dim \text{CoAmb}_Q^{(r)}$  many equations of shape

$$(11) \quad \text{linear combinations of } p_{\iota_1, \dots, \iota_r}^{[s]} \text{'s} = \text{nonlinear expressions of } p_{\kappa_1, \dots, \kappa_h}^{[s]} \text{'s } (h < r)$$

( $r \geq 2$ ), such that the linear terms on the left are themselves linearly independent formally, hence leading to the eliminability of a set of  $p_{i_1, \dots, i_r}^{[s]}$ 's of cardinality  $\dim \text{CoAmb}_Q^{(r)}$ . (If  $(\Psi^{(r)}/Q)_{\text{split}} = \left[ \phi_1^{(r)} \quad \phi_2^{(r)} \quad \phi_3^{(r)} \quad \phi_4^{(r)} \quad \phi_5^{(r)} \quad \phi_6^{(r)} \quad \phi_7^{(r)} \quad \phi_8^{(r)} \right]$ , then the equations above are restrictive equations for  $\phi_1^{(r)}, \phi_2^{(r)}, \phi_4^{(r)} - \phi_8^{(r)}, \phi_5^{(r)}, \phi_7^{(r)}$ .)

More precisely, equations (10) up to  $r$  are equivalent to  $(\Psi \odot \Psi)^{\leq r} = (-1, 0, -1)$ , i. e. Clifford productivity up to order  $r$ .

*Proof.* Indeed, in the light of the proof of the preceding theorem, involutivity means exactly a determinacy of the coambiguity terms in  $\text{vFQ}_Q^{(r)} = \text{Amb}_Q^{(r)} \oplus \text{CoAmb}_Q^{(r)}$ . This can be written down by recovering the  $U^{(i)}$ 's, yielding  $\bar{\pi}_Q^{(r)} \Psi^{(r)} = \bar{\pi}_Q^{(r)} (\Psi^{(r-1)})^{(r)}$ , i. e.

$$(12) \quad \bar{\pi}_Q^{(r)} \Psi^{(r)} = \text{something in terms of } \Psi^{\leq r-1}.$$

However, the exact shape on the right is not particularly straightforward. On the other hand, the identity  $\Psi \odot \Psi = (-1, 0, -1)$  implies  $(\Psi \odot \Psi)^{(r)} = 0$ , which implies  $\bar{\pi}_Q^{(r)} (\Psi \odot \Psi)^{(r)} = 0$ , which can be written as (10). Equation (10) is informative to the same degree as (12), yet it cannot add more information to it (i. e. to Clifford productivity, meant relative to the restrictive equations of lower order); hence they must be equivalent.  $\square$

**6.7. Remark.** (a) In (10), we have used only  $\bar{\pi}_Q^{(r)}|_{\text{PreCoAmb}_Q^{(r)}}$  essentially. Indeed, if we have the equations up to  $r-1$ , then  $(\Psi^{\leq r-1} \odot \Psi^{\leq r-1})^{(r)} \in \text{PreCoAmb}_Q^{(r)}$ . (This is transparent from writing  $\Psi^{\leq r-1} = (\Psi^{(r)})^{\leq r} + \Delta \Psi_{r-1}$ .)

(b) The arguments above have a version in the natural / conjugation-invariant case. Here one can assume that  $(Q_1, Q_2)$  is the Gram-Schmidt orthogonalization of  $(A_1, A_2)$  and  $\phi_1^{(r)}, \dots, \phi_8^{(r)}$  are  $r$ -homogeneous polynomials of  $r_1, r_2, r_4 = -r_8, r_5, r_7$ . (That is, the spaces  $\text{vFQ}_Q, \dots$  are replaced by  $\text{vFQ}_{Q_{\text{GS}}}, \dots$ ) Everything is completely analogous, except the expansions are in a reduced set of variables  $r_1, r_2, r_4 = -r_8, r_5, r_7$ . Or, alternatively, we can assume that  $(Q_1, Q_2)$  is the symmetric orthogonalization of  $(A_1, A_2)$  and we deal with polynomials of, say,  $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5$ ; etc.

(c) The setup above is also consistent with symmetry / orthogonal invariance, then  $U^{(r)}$ 's will be symmetric / orthogonal invariant. Using the exponential form one can make the setting compatible to transposition invariance.

**6.8.** Similar conclusions can be drawn regarding other kinds of Clifford productivity:

If  $\Psi$  is a Clifford conservative, Clifford productive, pseudoscalar FQ operation, then

$$\begin{aligned} \Psi(A_1, A_2) &= \dots (1 + U^{(3)})(1 + U^{(2)})(1 + U^{(1)}) \cdot Q_1 Q_2 \cdot \\ &\quad \cdot (1 + U^{(1)})^{-1} (1 + U^{(2)})^{-1} (1 + U^{(3)})^{-1} \dots, \end{aligned}$$

with  $U^{(r)} \in \text{sFQ}_Q^{(r)}$  and  $(U^{(r)})_{Q_1 Q_2}^0 = 0$ .

If  $\Psi$  is a Clifford conservative, floating Clifford productive FQ operation, then

$$\begin{aligned} \Psi(A_1, A_2) &= \dots (1 + U_1^{(3)})(1 + U_1^{(2)})(1 + U_1^{(1)}) \cdot (Q_1, Q_2) \cdot \\ &\quad \cdot (1 + U_2^{(1)})^{-1} (1 + U_2^{(2)})^{-1} (1 + U_2^{(3)})^{-1} \dots, \end{aligned}$$

with  $U_1^{(r)}, U_2^{(r)} \in \text{sFQ}_Q^{(r)}$  and  $(U_1^{(r)})_{Q_1 Q_2}^0 = Q_1 (U_2^{(r)})_{Q_1 Q_2}^0 Q_1^{-1}$ .

Again, we can derive the existence of the well-layered restrictive equations; the details are left to the reader.

## 7. INVOLUTIVITY AND IDEMPOTENCE

7.1. Typically, we want vectorial FQ operations satisfy one of the following properties:

(Inv) Involutivity: A vectorial FQ operation  $\Xi$  is an involution if  $\Xi \circ \Xi = \text{Id}$ .

(Idm) Idempotence :A vectorial FQ operation  $\Xi$  is idempotent, if  $\Xi \circ \Xi = \Xi$ .

(I3) 3-Idempotence: A vectorial FQ operation  $\Xi$  is 3-idempotent, if  $\Xi \circ \Xi \circ \Xi = \Xi$ .

Again, it is reasonable restrict to Clifford conservative operations.

7.2. Suppose that  $\Psi$  is a vectorial FQ operation. Using its expansion in the split base, we set

$$D\Psi := \left( \begin{bmatrix} p_1^{[1]} & p_5^{[1]} \\ p_1^{[2]} & p_5^{[2]} \end{bmatrix}, \begin{bmatrix} p_2^{[1]} & p_6^{[1]} \\ p_2^{[2]} & p_6^{[2]} \end{bmatrix}, \begin{bmatrix} p_3^{[1]} & p_7^{[1]} \\ p_3^{[2]} & p_7^{[2]} \end{bmatrix}, \begin{bmatrix} p_4^{[1]} & p_8^{[1]} \\ p_4^{[2]} & p_8^{[2]} \end{bmatrix} \right).$$

We call this the first differential of  $\Psi$ . By direct computation, it is easy to see

7.3. **Lemma.** *Suppose that  $\Psi_1, \Psi_2$  are Clifford conservative, vectorial FQ operations. Then*

$$D(\Psi_2 \circ \Psi_1) = D\Psi_2 \cdot D\Psi_1,$$

where the point the right side means multiplication componentwise.  $\square$

In particular, in order to have involutive or idempotent FQ operations in this setting, the components of  $D\Psi$  must be involutions or idempotents, respectively.

7.4. **Corollary.** *Suppose that  $\Psi$  is a transposition invariant and orthogonal invariant natural vectorial FQ operation.*

(a) *If  $\Psi$  is involutive, then  $\hat{p}_1^{[1]} = \hat{p}_2^{[1]}, \hat{p}_3^{[1]}, \hat{p}_4^{[1]} = \hat{p}_5^{[1]} \in \{-1, 1\}$ .*

(b) *If  $\Psi$  is idempotent, then  $\hat{p}_1^{[1]} = \hat{p}_2^{[1]}, \hat{p}_3^{[1]}, \hat{p}_4^{[1]} = \hat{p}_5^{[1]} \in \{0, 1\}$ .*

(c) *If  $\Psi$  is 3-idempotent, then  $\hat{p}_1^{[1]} = \hat{p}_2^{[1]}, \hat{p}_3^{[1]}, \hat{p}_4^{[1]} = \hat{p}_5^{[1]} \in \{-1, 0, 1\}$ .*

*Proof.* Direct computation in the mixed base.  $\square$

In these cases we have 8, 8 and 27 choices up to order 1; which we consider as the principal types for involutions, idempotents, and 3-idempotents, respectively.

7.5. Consider now expansions around a fixed Clifford system  $(Q_1, Q_2)$ . Let  $\mathcal{E}$  be expansion giving the Clifford system itself. For  $L \in \oplus^4 M_2(\mathbb{R})$ , let  $\mathcal{T}(L)$  be the expansion which is purely of order 1 and  $D(\mathcal{T}(L)) = L$ . Then, cf. the previous lemma,

$$(\mathcal{E} + \mathcal{T}(L_2)) \circ (\mathcal{E} + \mathcal{T}(L_1)) = (\mathcal{E} + \mathcal{T}(L_1 L_2)).$$

Let  $\Lambda \in \text{vFQ}^{(r)}$ , such that  $r \geq 1$ . Then in the compositions

$$(\mathcal{E} + \mathcal{T}(L)) \circ (\mathcal{E} + \Lambda) = \mathcal{E} + G_L(\Lambda)$$

and

$$(\mathcal{E} + \Lambda) \circ (\mathcal{E} + \mathcal{T}(L)) = \mathcal{E} + H_L(\Lambda),$$

the terms  $G_L(\Lambda)$  and  $H_L(\Lambda)$  are also from  $\text{vFQ}^{(r)}$ .

7.6. **Lemma.** *The actions  $G_{L_1}$  and  $H_{L_2}$  commute with each other. In fact, there is a two-sided associative action  $\oplus^4 M_2(\mathbb{R})$  on  $\text{vFQ}$ , such that  $G_{L_1}$  is a linear right action,  $H_{L_2}$  is a linear left action; moreover,  $G$  is also linear in  $L_1$ .*

*Proof.* This is the consequence of the associativity of the composition.  $\square$

7.7. **Theorem.** *Suppose that  $\Psi$  is a Clifford conservative, vectorial FQ operation. Then  $\Psi$  is invertible if and only if  $D\Psi$  is invertible.*

*Proof.*  $\mathcal{E} + \mathcal{T}((D\Psi)^{-1})$  will be an inverse modulo  $\text{vFQ}^{\geq 2}$ . However, one can easily see that if  $\Phi$  is inverse to  $\Psi$  modulo  $\text{vFQ}^{\geq r}$ , such that  $r \geq 2$ , then

$$\Phi + H_{(D\Psi)^{-1}}(\text{Id} - \Phi \circ \Psi)$$

is inverse to  $\Psi$  modulo  $\text{vFQ}^{\geq r+1}$ . Taking successive iterations, we obtain an inverse.  $\square$

**7.8. Lemma.** *Suppose that  $L \in \oplus^4 M_2(\mathbb{R})$  is an involution. Let*

$$\begin{aligned} \lambda_L^{(r)} &:= (H_L - G_L)|_{\text{vFQ}^{(r)}}, & \bar{\lambda}_L^{(r)} &:= (H_L + G_L)|_{\text{vFQ}^{(r)}}, \\ \varkappa_L^{(r)} &:= \frac{1}{4}(H_L - G_L)|_{\text{vFQ}^{(r)}}, & \bar{\varkappa}_L^{(r)} &:= \frac{1}{4}(H_L + G_L)|_{\text{vFQ}^{(r)}}. \end{aligned}$$

*Then, we claim, there are equalities*

$$\text{Amb}_L^{(r)} := \text{im } \lambda_L^{(r)} = \text{im } \varkappa_L^{(r)} = \ker \bar{\lambda}_L^{(r)} = \ker \bar{\varkappa}_L^{(r)}$$

*and*

$$\text{CoAmb}_L^{(r)} := \text{im } \bar{\lambda}_L^{(r)} = \text{im } \bar{\varkappa}_L^{(r)} = \ker \lambda_L^{(r)} = \ker \varkappa_L^{(r)}.$$

*Moreover,  $\lambda_L^{(r)}$  and  $\varkappa_L^{(r)}$  induce inverse bijections on  $\text{Amb}_L^{(r)}$ ; and  $\bar{\lambda}_L^{(r)}$  and  $\bar{\varkappa}_L^{(r)}$  induce inverse bijections on  $\text{CoAmb}_L^{(r)}$ . This leads to a direct decomposition*

$$\text{vFQ}^{(r)} = \text{Amb}_L^{(r)} \oplus \text{CoAmb}_L^{(r)}$$

*with projection factors*

$$\pi_L^{(r)} = \varkappa_L^{(r)} \circ \lambda_L^{(r)} = \lambda_L^{(r)} \circ \varkappa_L^{(r)} = \frac{1}{2}(\text{id} - H_L G_L)|_{\text{vFQ}^{(r)}}$$

*and*

$$\bar{\pi}_L^{(r)} = \bar{\varkappa}_L^{(r)} \circ \bar{\lambda}_L^{(r)} = \bar{\lambda}_L^{(r)} \circ \bar{\varkappa}_L^{(r)} = \frac{1}{2}(\text{id} + H_L G_L)|_{\text{vFQ}^{(r)}}.$$

*Proof.* This follows from that  $G_L$  and  $H_L$  are commuting linear actions with spectrum in  $\{-1, 1\}$ .  $\square$

**7.9. Theorem.** *Suppose that  $\Psi$  is an involutive, Clifford conservative, vectorial FQ operation with  $D\Psi = L$ . Then*

$$(13) \quad \begin{aligned} \Psi &= \dots \circ (\text{Id} + U^{(4)}) \circ (\text{Id} + U^{(3)}) \circ (\text{Id} + U^{(2)}) \circ (\mathcal{E} + \mathcal{T}(L)) \circ \\ &\quad \circ (\text{Id} + U^{(2)})^{-1} \circ (\text{Id} + U^{(3)})^{-1} \circ (\text{Id} + U^{(4)})^{-1} \circ \dots, \end{aligned}$$

*where  $U^{(i)} \in \text{vFQ}^{(i)}$ . Furthermore, we can obtain a most economical choice for the  $U^{(r)}$  (yielding bijective correspondence to  $\Psi$ ) by choosing  $U^{(r)}$  from  $\text{Amb}_L^{(r)}$ .*

*With this choice,  $U^{(r)}$  can be recovered from  $\Psi / \text{vFQ}^{\geq r+1}$  by simple arithmetics.*

*Proof.* First, on the constructive side: One can see that with arbitrary choice, (13) yields an involutive operation  $\Psi$ . (Because it makes sense and it behaves so in every order.) Let  $\Psi^{(r)}$  be the version of the FQ operation when we use only  $U^{(2)}, \dots, U^{(r)}$ , but not further. Then

$$\Psi^{(r)} - \Psi^{(r-1)} = \Delta\Psi^{(r)} + O((R/Q)^{\geq r+1}),$$

where  $\Delta\Psi^{(r)}$  is  $r$ -homogeneous in  $(R/Q)$ . In fact,

$$\Delta\Psi^{(r)} = H_L(U^{(r)}) - G_L(U^{(r)}).$$

This term describes an ambiguity which arises we step up from order  $r-1$  to  $r$ . These possible terms form  $\text{Amb}_L^{(r)}$ . Furthermore, if an ambiguity  $V^{(r)} = \Delta\Psi^{(r)} \in \text{Amb}_L^{(r)}$  is given, then it is induced by  $U^{(r)} = \varkappa_L^{(r)}(V^{(r)}) \in \text{Amb}_L^{(r)}$ . We also see that different choices of  $U^{(r)} \in \text{Amb}_L^{(r)}$  lead to different ambiguities  $V^{(r)}$ .

However, one cannot do better even in the general case. Suppose that we an Clifford conservative operation FQ operation from  $\text{vFQ}^{\leq r-1}$ , with first differential  $L$ , so its square is in  $\text{Id} + \text{vFQ}^{\geq r}$ , i. e. it is involutive modulo  $\text{vFQ}^{\geq r}$ . Suppose that we manage it extend it to an FQ operation  $\Psi \in \text{vFQ}^{\leq r}$ , involutive modulo  $\text{vFQ}^{\geq r+1}$ , so its square is in  $\text{Id} + \text{vFQ}^{\geq r+1}$ . But there is the possibility of getting another version  $\Psi'$  so that

$$\Psi' = \Psi + V^{(r)} + O((R/Q)^{\geq r+1}),$$

where  $V^{(r)} \in \text{vFQ}^{(r)}$ . Then  $\Psi' \circ \Psi' \in \text{Id} + \text{vFQ}^{\geq r+1}$  means that

$$H_L(V^{(r)}) + G_L(V^{(r)}) = 0,$$

i. e.,  $V^{(r)} \in \text{Amb}_L^{(r)}$ . Proceeding inductively, we see that every involutive Clifford conservative, vectorial FQ operation with first differential  $L$  occurs in form (13), and in fact, using the economical choice for the  $U^{(r)}$ . Then, in each step, we have a full involutive operation  $\Psi^{(r)}$ , and the modifier terms arithmetically determined as follows: If  $U^{(2)}, \dots, U^{(r-1)}$  are already recovered, then  $U^{(r)} = \varkappa_L^{(r)}(\Psi - (\Psi^{(r-1)})^{(r)})$ .  $\square$

**7.10. Corollary.** *Suppose that  $\Psi_1, \Psi_2$  are involutive, Clifford conservative, vectorial FQ operations. Then,  $D\Psi_1$  and  $D\Psi_2$  are conjugates of each other if and only if  $\Psi_1$  and  $\Psi_2$  are conjugates of each other by Clifford conservative FQ operations.*

*Proof.* This is a consequence of the previous theorem and Lemma 7.3.  $\square$

**7.11. Theorem.** *Consider vectorial FQ operations  $\Psi$ , with properties such that  $\Psi$  is Clifford conservative and  $D\Psi = L$  is an involution.*

*Then, we claim, the involutivity of  $\Psi$  can be expressed by*

$$(14) \quad \bar{\lambda}_L^{(r)} \Psi^{(r)} + \bar{\pi}_L^{(r)}(\Psi^{\leq r-1} \circ \Psi^{\leq r-1})^{(r)} = 0,$$

or

$$(15) \quad \bar{\pi}_L^{(r)} \Psi^{(r)} + \bar{\varkappa}_L^{(r)}(\Psi^{\leq r-1} \circ \Psi^{\leq r-1})^{(r)} = 0$$

( $r \geq 2$ ). That is, a collection of  $\dim \text{CoAmb}_L^{(r)}$  many equations of shape

$$(16) \quad \text{linear combinations of } p_{i_1, \dots, i_r}^{[s]} \text{'s} = \text{nonlinear expressions of } p_{\lambda_1, \dots, \lambda_h}^{[s]} \text{'s } (h < r)$$

( $r \geq 2$ ), such that the linear terms on the left are themselves linearly independent formally hence leading to the eliminability of a set of  $p_{i_1, \dots, i_r}^{[s]}$ 's of cardinality  $\dim \text{CoAmb}_L^{(r)}$ .

More precisely, equations (15)/(16) up to  $r$  are equivalent to  $(\Psi \circ \Psi - \text{Id})^{\leq r} = 0$ , i. e. involutivity up to order  $r$ .

*Proof.* Indeed, in the light of the proof of the preceding theorem, involutivity means exactly a determinacy of the coambiguity terms in  $\text{vFQ}^{(r)} = \text{Amb}_L^{(r)} \oplus \text{CoAmb}_L^{(r)}$ . This can be written down by recovering the  $U^{(i)}$ 's, yielding  $\bar{\pi}_L^{(r)} \Psi^{(r)} = \bar{\pi}_L^{(r)}(\Psi^{(r-1)})^{(r)}$ , i. e.

$$(17) \quad \bar{\pi}_L^{(r)} \Psi^{(r)} = \text{something in terms of } \Psi^{\leq r-1}.$$

However, the exact shape on the right is not particularly straightforward. On the other hand, from the identity  $\Psi \circ \Psi = \text{Id}$  shows

$$(G_L + H_L)(\Psi^{(r)}) + (\Psi^{\leq r-1} \circ \Psi^{\leq r-1})^{(r)} = 0,$$

which implies (14), which is equivalent to (15). Equation (15) is informative to the same degree as (17), yet it cannot add more information to it (i. e. to involutivity); hence they must be equivalent (that is relative to the restrictive equations of of lower order).  $\square$

**7.12. Remark.** (a) In (16), we have used only  $\bar{\kappa}_L^{(r)}|_{\text{CoAmb}_L^{(r)}}$  essentially. Indeed, if we have the equations up to  $r-1$ , then  $(\Psi^{\leq r-1} \circ \Psi^{\leq r-1})^{(r)} \in \text{CoAmb}_L^{(r)}$ . (This is transparent from writing  $\Psi^{\leq r-1} = (\Psi^{(r)})^{\leq r} + \Delta\Psi_r$ .)

(b) Furthermore, the line of arguments in this section have a version for conjugation-invariant FQ operations, where the expansions are relative, say, to the symmetric orthogonalizations, hence we can assume  $\Psi$  has an expansion in the variables  $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5$ . However, for the sake of compositions one should consider the conjugation-invariant extensions  $\Psi^{\text{ext}} = \Psi_{\mathcal{Q}^{\text{Sy}}}$ . In this setting  $\Psi_1 \circ \Psi_2$  should be replaced by  $\Psi_1^{\text{ext}} \circ \Psi_2$ .

Naturally, at the end, this leads to the same result as if we apply the restrictive equations to more restricted arguments (whose symmetric orthogonalization is  $(Q_1, Q_2)$ ).

(c) Again, these arguments are compatible to symmetry or orthogonal invariance.

**7.13. Example.** Consider the case  $\Psi$  is symmetric Clifford conservative, involutive FQ operation with

$$\hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

In order to describe this case, let us consider the parity twist automorphism  $\text{ptw} : \mathfrak{C} \rightarrow \mathfrak{C}$  given by

$$\begin{array}{c|cccccccc} x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \text{ptw}(x) & 2 & 1 & 3 & 4 & 5 & 6 & 8 & 7 \end{array};$$

and the indicator function  $\text{ind} : \mathfrak{C} \rightarrow \{1, -1\}$

$$\begin{array}{c|cccccccc} x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \text{ind}(x) & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{array}.$$

The expansion orders 0 and 1 are determined, and symmetry implies that it is sufficient to consider  $\hat{p}_{\iota_1, \dots, \iota_r}^{[1]}$ . Beyond that, in the mixed base, the involutivity can be expressed by the three sets of equations ( $r \geq 2$ )

$$(I) \quad \hat{p}_{\iota_1, \dots, \iota_r}^{[1]} = \text{expression of } \hat{p}_{\text{lesser that } r \text{ many indices}}^{[1]} \text{'s}$$

where  $\iota_1 * \dots * \iota_r \in \{4, 5\}$ , and  $\prod_{j=1}^r \text{ind}(\iota_j) = -1$ ;

$$(II) \quad \hat{p}_{\iota_1, \dots, \iota_r}^{[1]} = \text{expression of } \hat{p}_{\text{lesser that } r \text{ many indices}}^{[1]} \text{'s}$$

where  $\{\iota_1, \dots, \iota_r\} \subset \{3, 4, 5, 6\}$ ,  $\iota_1 * \dots * \iota_r \in \{3, 6\}$ , and  $\text{ind}(\iota_1 * \dots * \iota_r) \prod_{j=1}^r \text{ind}(\iota_j) = 1$ ;

$$(III) \quad \hat{p}_{\iota_1, \dots, \iota_r}^{[1]} + \left( \text{ind}(\iota_1 * \dots * \iota_r) \prod_{j=1}^r \text{ind}(\iota_j) \right) \hat{p}_{\text{ptw}(\iota_1), \dots, \text{ptw}(\iota_r)}^{[1]} = \\ = \text{expression of } \hat{p}_{\text{lesser that } r \text{ many indices}}^{[1]} \text{'s}$$

where  $\{\iota_1, \dots, \iota_r\} \not\subset \{3, 4, 5, 6\}$ ,  $\iota_1 * \dots * \iota_r \in \{1, 2, 3, 6, 7, 8\}$ . In case (III) it can be assumed that  $(\iota_1, \dots, \iota_r)$  is lexicographically precedes  $(\text{ptw}(\iota_1), \dots, \text{ptw}(\iota_r))$ . (It would have also been a good strategy to introduce another base by “demixing”  $\{1, 2\}$  and  $\{7, 8\}$ .)

In the conjugation-invariant case, we are restricted to  $\{\iota_1, \dots, \iota_r\} \subset \{1, 2, 3, 4, 5\}$ ; and  $\prod_{j=1}^r \text{ind}(\iota_j)$  simplifies to  $(-1)^r$ .

**7.14.** When it comes to idempotent operations, the element  $L$  should be idempotent. In this setting, one should use

$$\begin{aligned} \lambda_L^{(r)} &= (H_L - G_L)|_{\text{vFQ}^{(r)}}, & \bar{\lambda}_L^{(r)} &= (H_L + G_L - \text{id})|_{\text{vFQ}^{(r)}}, \\ \kappa_L^{(r)} &= (H_L - G_L)|_{\text{vFQ}^{(r)}}, & \bar{\kappa}_L^{(r)} &= (H_L + G_L - \text{id})|_{\text{vFQ}^{(r)}}; \end{aligned}$$

with

$$\bar{\pi}_L^{(r)} = (\text{id} - H_L - G_L + 2H_L G_L)|_{\text{vFQ}^{(r)}}.$$

Then we have analogous arguments, leading to restrictive equations of shape as in (15)/(16).

In the case of 3-idempotents, one can proceed with

$$\lambda_L^{(r)} = (H_L - G_L)|_{\text{vFQ}^{(r)}}, \quad \bar{\lambda}_L^{(r)} = (H_L^2 + H_L G_L + G_L^2 - \text{id})|_{\text{vFQ}^{(r)}},$$

$$\varkappa_L^{(r)} = (H_L - G_L + \frac{3}{4}H_L^2 G_L - \frac{3}{4}H_L G_L^2)|_{\text{vFQ}^{(r)}},$$

$$\bar{\varkappa}_L^{(r)} = (-\frac{3}{4}H_L^2 G_L^2 + \frac{1}{4}H_L G_L + G_L^2 + H_L^2 - \text{id})|_{\text{vFQ}^{(r)}};$$

with

$$\bar{\pi}_L^{(r)} = (\text{id} - H_L^2 - G_L^2 + \frac{3}{2}H_L^2 G_L^2 + \frac{1}{2}H_L G_L)|_{\text{vFQ}^{(r)}}.$$

The shape of the restrictive equations is

$$\bar{\lambda}_L^{(r)} \Psi^{(r)} + \bar{\pi}_L^{(r)} (\Psi^{\leq r-1} \circ \Psi^{\leq r-1} \circ \Psi^{\leq r-1})^{(r)} = 0,$$

or

$$\bar{\pi}_L^{(r)} \Psi^{(r)} + \bar{\varkappa}_L^{(r)} (\Psi^{\leq r-1} \circ \Psi^{\leq r-1} \circ \Psi^{\leq r-1})^{(r)} = 0;$$

otherwise, the conclusions are similar.

**7.15. Corollary.** *Suppose that  $\Psi$  is natural, orthogonal invariant, Clifford conservative vectorial FQ operation. Then  $\Psi$  is Clifford conservative if and only if  $\Psi^{\leq 1} = (\mathcal{O}^{\text{Sy}})^{\leq 1}$  and  $\Psi$  is idempotent.*

*Proof.* In either way, we have a conjugate of  $\mathcal{O}^{\text{Sy}}$ , preserving the noted properties.  $\square$

**7.16. Example.** Consider the case when  $\Psi$  is natural, symmetric Clifford conservative, Clifford productive FQ operation. Then, in the mixed base,

$$\hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The expansion orders 0 and 1 are determined, symmetry implies that it is sufficient to consider  $\hat{p}_{\iota_1, \dots, \iota_r}^{[1]}$ , and naturality / conjugation invariance implies that it is sufficient to consider  $\{\iota_1, \dots, \iota_r\} \subset \{1, 2, 3, 4, 5\}$ .

Beyond that, the Clifford productivity / idempotence can be expressed by three sets of equations ( $r \geq 2$ )

$$(I) \quad \hat{p}_{\iota_1, \dots, \iota_r}^{[1]} = \text{expression of } \hat{p}_{\text{lesser that } r \text{ many indices}}^{[1]} \text{'s}$$

where  $\iota_1 * \dots * \iota_r \in \{4, 5\}$ ;

$$(II) \quad \hat{p}_{\iota_1, \dots, \iota_r}^{[1]} = \text{expression of } \hat{p}_{\text{lesser that } r \text{ many indices}}^{[1]} \text{'s}$$

where  $\{\iota_1, \dots, \iota_r\} \subset \{3, 4, 5\}$ ,  $\iota_1 * \dots * \iota_r \in \{3\}$ ;

$$(III) \quad \hat{p}_{\iota_1, \dots, \iota_r}^{[1]} - \text{ind}(\iota_1 * \dots * \iota_r) \hat{p}_{\text{ptw}(\iota_1), \dots, \text{ptw}(\iota_r)}^{[1]} = \text{expression of } \hat{p}_{\text{lesser that } r \text{ many indices}}^{[1]} \text{'s}$$

where  $\{\iota_1, \dots, \iota_r\} \not\subset \{3, 4, 5\}$ ,  $\iota_1 * \dots * \iota_r \in \{1, 2, 3, 6, 7, 8\}$ .

## 8. AN EXAMPLE OF APPLICATION OF RESTRICTIVE EQUATIONS

8.1. Suppose that we have some of arithmetical conditions  $\mathcal{S}$ ; and we consider those say, vectorial, FQ operations which satisfy them modulo  $\text{vFQ}^{\geq r+1}$ . This yields a subvariety  $V_r \subset \text{vFQ}^{\leq r}$ . Taking further restrictions in the expansion we have a sequence of subsets

$$\{*\} \xleftarrow{\theta_0} V_r^{\leq 0} \xleftarrow{\theta_1} V_r^{\leq 1} \xleftarrow{\theta_2} \dots \xleftarrow{\theta_r} V_r^{\leq r} = V_r.$$

We will consider only cases when

(AFP) the fibers  $\theta_i$  are affine linear spaces of constant dimension

of  $d_r^{(i)} = \dim(\theta_i)^{-1}(x)$ ,  $x \in V_r^{\leq i-1}$  (affine linear fiber property). We can collect the dimension data into a table

	0	1	2	3	...
0	$d_0^{(0)}$				
1	$d_1^{(0)}$	$d_1^{(1)}$			
2	$d_2^{(0)}$	$d_2^{(1)}$	$d_2^{(2)}$		
3	$d_3^{(0)}$	$d_3^{(1)}$	$d_3^{(2)}$	$d_3^{(3)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

If  $j$  is fixed, then  $d_r^{(j)}$  is monotone decreasing as  $r \rightarrow \infty$ . One can easily see that there are two cases: The process either runs into inconsistency at some level, or, it converges

$$\lim_{r \rightarrow \infty} d_r^{(j)} = d_{\mathcal{S}}^{(j)},$$

to a sequence of  $d_{\mathcal{S}}^{(j)}$ 's. If the affine linear fiber property holds out, then one can show that this leads to a possibly infinite-dimensional filtered variety  $V_{\mathcal{S}} \subset \text{vFQ}$  with fiber dimensions  $d_{\mathcal{S}}^{(j)}$ . These numbers tell us the degree of freedom  $\mathcal{S}$  allows on the  $j$ th expansion level. The most advantageous case is when  $d_{\mathcal{S}}^{(j)} = 0$  for all  $j$ ; this means that the vectorial operation is completely characterized by  $\mathcal{S}$ .

Carrying out a complete analysis along these lines is, in general, is very difficult (although occasionally works out, cf. Theorems 4.6 and 4.8); but doing it up to a finite degree often serves us with useful lessons and ideas. The following provides an illustration.

8.2. Consider the following situation: We know the FQ orthogonalization procedure  $\mathcal{O}^{\text{Sy}}$ , but we are dissatisfied by it, because its “axis”  $\mathcal{D} \circ \mathcal{O}^{\text{Sy}}$  (a pseudoscalar FQ operation) is not affine scaling invariant. So, we are looking for a nice FQ orthogonalization procedure  $\Psi$  with better properties.

(i) We require that  $\Psi$  should be Clifford conservative, transposition invariant, orthogonal invariant, and Clifford productive. The collection of these properties implies the fiber dimensions

	0	1	2	3	4	5
0	0					
1	0	0				
2	0	0	2			
3	0	0	2	12		
4	0	0	2	12	56	
5	0	0	2	12	56	270

So, it seems, there are many operations like that. (We know that imposing metric trace commutativity, or the fiber-star property, we would obtain  $\mathcal{O}^{\text{Sy}}$ ; but we have chosen an other way.)



(ii) Next, we impose that the pseudoscalar FQ operation  $\mathcal{D} \circ \Psi$  should satisfy the  $\hat{r}_i$ -scaling invariances with  $i \in \{1, 2, 3, 4, 5\}$ , with  $\alpha = 0$  (among them is affine invariance,  $i = 3, 4$ ). We have fiber dimensions

	0	1	2	3	4	5
0	0					
1	0	0				
2	0	0	1			
3	0	0	0	5		
4	0	0	0	5	22	
5	0	0	0	5	16	109

We see that the various conditions, well-layered in themselves, interact with each other, leading to further constraints in *lower* than top expansion degrees (“undercut”).

(iii) Encouraged by the possibilities, next we simply declare that

$$\mathcal{D} \circ \Psi = \underline{\mathcal{A}}_C,$$

where the central axis operation  $\underline{\mathcal{A}}_C$  is defined by

$$(18) \quad \underline{\mathcal{A}}_C(A_1, A_2) = \text{pol } \frac{1}{2}(\underline{\mathcal{A}}_L(A_1, A_2) + \underline{\mathcal{A}}_R(A_1, A_2)).$$

(Assuming that we know from somewhere that  $\underline{\mathcal{A}}_C$  is sufficiently nice for that purpose.) This leads to fiber dimensions

	0	1	2	3	4	5
0	0					
1	0	0				
2	0	0	0			
3	0	0	0	4		
4	0	0	0	4	16	
5	0	0	0	4	16	92

(iv) Next, we impose the property

$$(19) \quad ([A_1, \Psi(A_1, A_2)]_1 + [A_2, \Psi(A_1, A_2)]_2)_{\underline{\mathcal{A}}_C(A_1, A_2)}^0 = 0,$$

a weakened version of metric trace commutativity. This leads to fiber dimensions

	0	1	2	3	4	5
0	0					
1	0	0				
2	0	0	0			
3	0	0	0	0		
4	0	0	0	0	0	
5	0	0	0	0	0	0

which is quite encouraging to think that we have characterized a certain FQ operation  $\mathcal{O}^{\text{Sy}, \mathcal{A}_C}$ . But this is not a proof yet. However, using (CP) and (18), we can rearrange (19), as

$$[(A_1)_{\underline{\mathcal{A}}_C(A_1, A_2)}^1, \Psi(A_1, A_2)]_1 + [(A_2)_{\underline{\mathcal{A}}_C(A_1, A_2)}^1, \Psi(A_1, A_2)]_2 = 0,$$

from which we can deduce

$$\mathcal{O}^{\text{Sy}, \mathcal{A}_C}(A_1, A_2) = \Psi(A_1, A_2) = \mathcal{O}^{\text{Sy}}((A_1)_{\underline{\mathcal{A}}_C(A_1, A_2)}^1, (A_2)_{\underline{\mathcal{A}}_C(A_1, A_2)}^1).$$

Here, the good properties of  $\mathcal{O}^{\text{Sy}, \mathcal{A}_C} = \Psi$  defined above are immediate: Indeed,  $\mathcal{D} \circ \Psi$  will be a Clifford conservative, Clifford productive pseudoscalar operation multiplicatively

commuting with the similar operation  $\mathcal{A}_C$ , from which one can derive that they are equal. So, in retrospect, this is quite simple.

(iv') On the other hand, we may look for an other kind of operation by requiring compatibility with  $\mathcal{O}^{\text{afSy}}$  and  $\mathcal{O}^{\text{fSy}}$ ; so, instead of (19),

$$(20) \quad \Psi \circ \mathcal{O}^{\text{fSy}} = \Psi \quad \text{and} \quad \Psi \circ \mathcal{O}^{\text{afSy}} = \Psi$$

should be satisfied. (Unfortunately,  $\mathcal{O}^{\text{Sy}, \mathcal{A}_C}$  fails (20); and  $\mathcal{O}^{\text{Sy}} \circ \mathcal{O}^{\text{fSy}}$  and  $\mathcal{O}^{\text{Sy}} \circ \mathcal{O}^{\text{afSy}}$  fail (18).) In this case, the first few fiber dimensions are

	0	1	2	3	4	5
0	0					
1	0	0				
2	0	0	0			
3	0	0	0	0		
4	0	0	0	0	3	
5	0	0	0	0	3	0

leaving the case quite open. In fact, one can construct a relatively nice operation (axial orthogonalization) satisfying the requirements, but which we do not explain here.

8.3. At this point, the reader may wonder on the following:

(i) This “method” seems to be much more suitable to prove non-existence than existence. This is true, indeed. On the other hand, in practice, most results *are* negative. It is just very easy to make wrong guesses about FQ operations. In the author’s experience, FQ operations almost always fail to be so nice as one would like them to be, but do not fail to stay a little bit mysterious. So, in fact, this method serves well our analytical efforts.

(ii) It is not clear what is the exact relevance of the previous sections in this method, in general. Indeed, as computations are tedious, sooner or later one should impose non-basic invariance rules, and one resorts to using computers anyway; the exact form of the restrictive equations gets irrelevant. This is also true. On the other hand, in a situation where data grows exponentially in the expansion order, *any* edge in the computation is welcome. One should avoid solving large systems of equations as much as possible. In the author’s experience, regarding FQ operations, one cannot really obtain a correct picture just from extrapolating from a couple of expansion orders (say  $r = 1, 2, 3$ ). In some relevant cases the first ambiguities appear in expansion orders  $r = 6, 7$ . Hence using appropriate bases, etc., makes a difference.

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